BACHELOR'S DEGREE THESIS

Degree in Mathematics

Title: The Selberg Trace Formula

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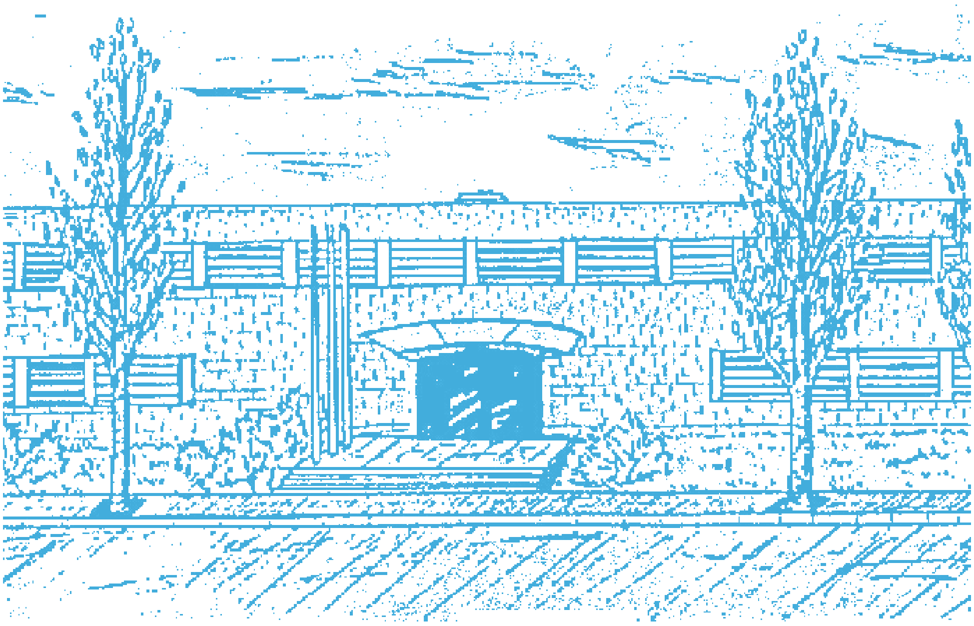
Academic year: 2015/16



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Bachelor's degree project

The Selberg Trace Formula

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June 27, 2016

Contents

Preface

This thesis presents a connection between Spectral Theory (in particular, the spectrum of the Laplace-Beltrami operator) and Riemannian geometry (in particular, the geometry of a symmetric three-dimensional space of constant negative curvature: the Hyperbolic space $\mathbb{H}$).

The Laplace-Beltrami operator is defined as the divergence of the gradient acting on functions defined in a Riemannian manifold. These operators are defined by a Riemannian metric g that is, a symmteric, positive-definite two-tensor. An important property of the Laplace-Beltrami operator is that commutes with the elements of the isometry group of the Riemannian manifold (see page 387 proposition 2.1 [14]). In this work a trace formula called The Selberg Trace Formula is obtained for any cocompact subgroup Γ of the isometry group of $\mathbb{H}$. This formula is given in [7], [9] in a more general case: when $\Gamma \setminus \mathbb{H}$ defines a non-compact but finite three-dimensional manifold and where the Laplace-Beltrami operator acts on vector functions Γ-automorphic (that is Γ-invariant) by a given representation χ. In this work we provide a proof for the case that $\Gamma \setminus \mathbb{H}$ is compact and where the Laplace-Beltrami operator acts on Γ-automorphic functions (non-vector functions). Most of the applications to Number Theory and Cosmology lie in the case we're not dealing with. The applications of the result we present are from Spectral Geometry.

In Chapter 1 we give an introduction of the developement of the trace formulas and the connection with the Riemann Hypothesis.

In Chapter 2 we introduce preliminary concepts about the geometry and the isometry group of $\mathbb{H}$. We start introducing the two-dimensional case providing almost all proofs, where the statements are given but proofs as exercices in [2]. Then, we state some results on the three-dimensional case (proving a few of them). The only background needed are some concepts on differential geometry.

In Chapter 3 is given a more technical and analytical part. We develop the functions that allow us to compute the trace via the resolvent of the Laplace-Beltrami operator. Almost all proofs are given, and the most technical proofs are those that we don't provide.

In Chapter 4 we compute the trace formula for any cocompact group Γ, subgroup of the isometry group of $\mathbb{H}$ and we show the direct applications of such formula. The most relevant is The Prime Geodesic Theorem which is related with the Prime Number Theorem.

In Chapter 5 we provide some symmetric functions defined in the spectral context and the connection between prime numbers and primitive geodesics.

I want to tank Joan Carles Lario for reading the thesis and for leading me to do this project I've chosen. I also thank María del Mar González for giving some ideas for the project.

Guillermo Aparicio Estrems. *Barcelona, Spain. June 2016.*

I developed my theory of infinitely many variables from purely mathematical interests, and even called it 'spectral analysis' without any presentiment that it would later find an application to the actual spectrum of physics!

David Hilbert (C. Reid. *Hilbert,* New York, 1996. P. 183)

1 Introduction

In 1956 Atle Selberg showed a formula connecting the spectrum of the Laplace operator and the geodesics of the manifold where the operator is acting to. The formula is called Selberg Trace Formula, similar in some sense as the Poisson summation formula. We can even say that the Selberg trace formula is an analog of the Poisson summation formula for compact manifolds of constant negative curvature. In this work we explain the trace formula in a more intuitive way but not in all its generality. To understand why does Selberg and other mathematicians were interested in this formula we start with the principal problem which Selberg was worried about (as a lot of mathematicians), the Riemann Hypothesis.

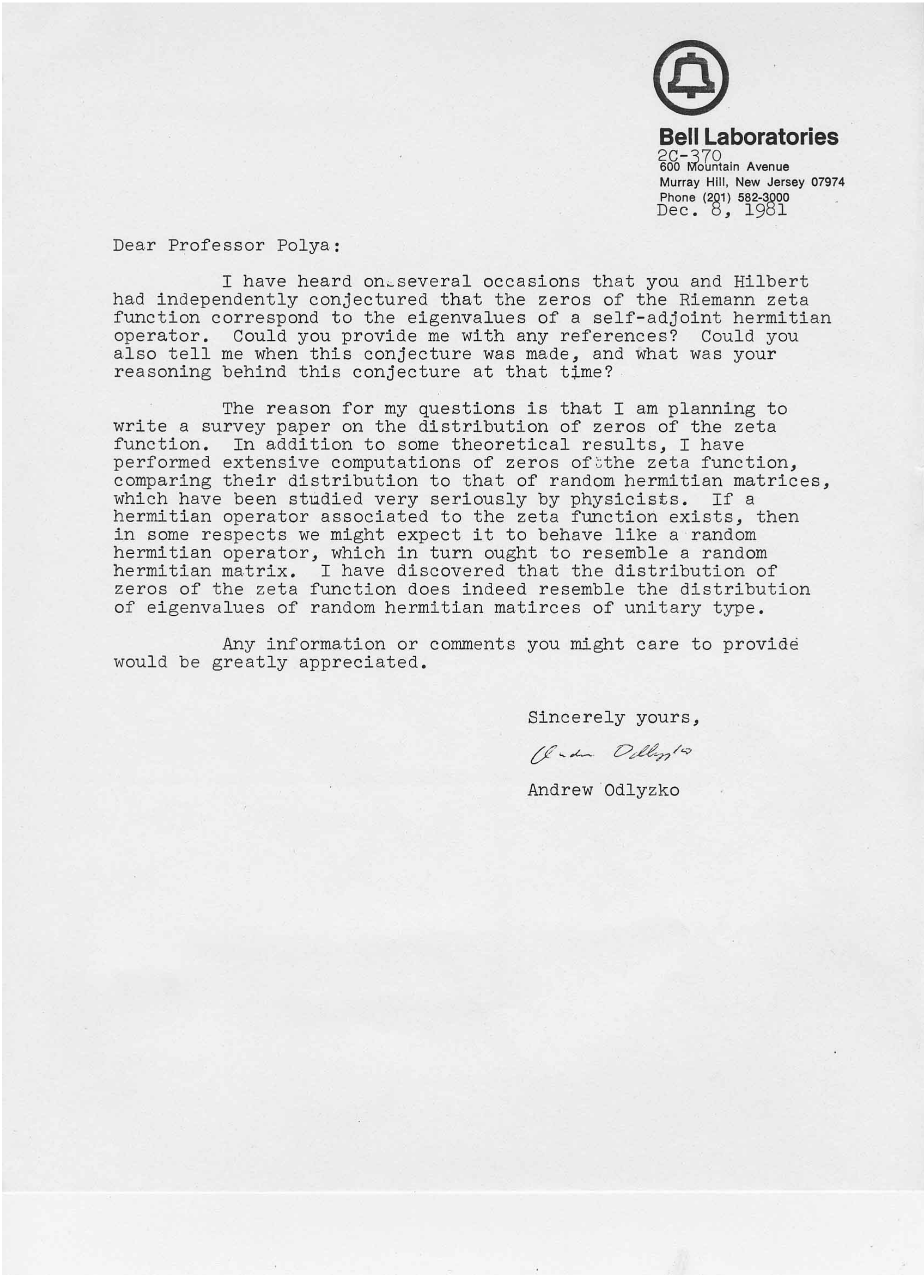
1.1 The Riemann Hypothesis

In 1859 Bernhard Riemann showed a property of a function studied years before by Leonhard Euler and called the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s \in \mathbb{C},\ \Re s > 1.$$

This function is a complex analytical function for $\Re s > 1$ and it can be extended analytically to all of $\mathbb{C}$ except at the simple pole $s = 1$ of $\zeta(s)$. The analitical extension of ζ (also denoted ζ) has the well-known trivial zeros $-2, -4, -6, ...$ What Riemann had conjectured is that all the non-trivial zeros (called the Riemann zeros [20]) lie in the line $\Re s = \frac{1}{2}$.

Calculating some Riemann zeros arises an idea stated by Hilbert and Pólya independently, some time in the 1910s [24]. This idea is given as a new conjuecture of the non-trivial zeros of ζ called the Hilbert-Pólya conjecture and states that such zeros are the spectrum of a self-adjoint operator. This would imply the Riemann Hypothesis. In the same paper Selberg published the trace formula ([26]) Selberg defines a zeta function (called the Selberg zeta function) which satisfies the Hilbert-Pólya conjecture. That is, the zeros of the Selberg zeta function are eigenvalues of a self-adjoint operator and this operator is the Laplace operator (the laplacian) defined over the hyperbolic half-plane (that is all $s \in \mathbb{C}:\ \Im s > 0$ where there is a Riemmannian metric defined over all the half-plane, which has constant negative curvature -1). Since the relation between the Selberg trace formula (which is a sum taken over the eigenvalues of the corresponding operator that is, the hyperbolic Laplace-Beltrami operator) and the Selberg zeta function is the logarithmic derivative the connection, between the Hilbert-Pólya conjecture and the Riemann Hypothesis is in some sense intuitive.



Bell Laboratories
2C-370
600 Mountain Avenue
Murray Hill, New Jersey 07974
Phone (201) 582-3000
Dec. 8, 1981

Dear Professor Polya:

I have heard on several occasions that you and Hilbert had independently conjectured that the zeros of the Riemann zeta function correspond to the eigenvalues of a self-adjoint hermitian operator. Could you provide me with any references? Could you also tell me when this conjecture was made, and what was your reasoning behind this conjecture at that time?

The reason for my questions is that I am planning to write a survey paper on the distribution of zeros of the zeta function. In addition to some theoretical results, I have performed extensive computations of zeros of the zeta function, comparing their distribution to that of random hermitian matrices, which have been studied very seriously by physicists. If a hermitian operator associated to the zeta function exists, then in some respects we might expect it to behave like a random hermitian operator, which in turn ought to resemble a random hermitian matrix. I have discovered that the distribution of zeros of the zeta function does indeed resemble the distribution of eigenvalues of random hermitian matirces of unitary type.

Any information or comments you might care to provide would be greatly appreciated.

Sincerely yours,

Andrew Odlyzko

Andrew Odlyzko

Figure 1: The first letter from Pólya appears to present the only documented evidence about the origins of the conjecture. The account of the formulation of the conjecture in the first letter is consistent with what Pólya had told Dennis Hejhal in a personal conversation [24].

1.2 Harmonic Analysis and Trace Formulas

Harmonic analisis is a branch of mathematics studied first by J. Fourier and is a generalization of the Fourier series and Fourier transforms. In the last century many applications to medicine ([28], [15]), geophysics ([28]), acoustics ([21]), aerospatial engineering ([21]), cosmology ([2]), quantum physics ([2]), statistics ([28]) and number theory ([28]) appeared. A more particular branch is harmonic analysis on symmetric spaces that is, adding some special properties to the geometry of the space where we are working on, to obtain more useful results (see [14]). Examples of such spaces are the Euclidean space, the sphere, the hyperbolic space (for any dimension $n \geq 1$). Harmonic analysis has a lot of connections with other fields in mathematics like Complex Analysis, Number Theory, Differential Geometry, Automorphic functions and Automorphic forms, Graph Theory, Algebraic Topology, Dynamical Systems, Algebraic Geometry, Functional Analysis, Partial Differential Equations, Fractional Analysis, Fractal Geometry...

In this work we develop the theory of harmonic analysis of the three-dimensional hyperbolic space. As in the sphere and the circle, one can derive a trace formula that is, a formula that connects the eigenvalues of the Laplace-Beltrami operator with the length of the geodesics (or other geometric values) of the manifold. In the case of the hyperbolic space (in all dimensions $n > 1$) there are none non-zero eigenvalues which are known so the trace formula is useful to derive some properties of the eigenvalues (asymptotic relations for example). It is interesting to study the symmetric functions of the eigenvalues called the theta, zeta and delta functions which gives the respective trace of the heat semigroup, trace of the complex power, and the characteristic determinant of the Laplace operator, the trace formula simplifies their computations. These functions are used to calculate physical quantities as the Cassimir energies in Quantum field theory and Quantum mechanics ([11], [16], [6]). These symmetric functions satisfy some analog properties to the Riemann zeta function and they generalize these properties in some sense. We will explain these concepts in the last chapter, the idea is to construct a geometric zeta function which has only zeros at the eigenvalues of the Laplace operator (and hence satisfies the Hilbert-Pólya conjecture) and then construct its superzeta function that is, the spectral zeta function (see [29]). Some questions arise: Can we give a geometric interpretation of the Riemann zeta function to derive that its superzeta function is a spectral function? Can we obtain an operator realted to the geometry information given by the Riemann zeta function to derive that the Riemann zeros are eigenvalues of such operator?

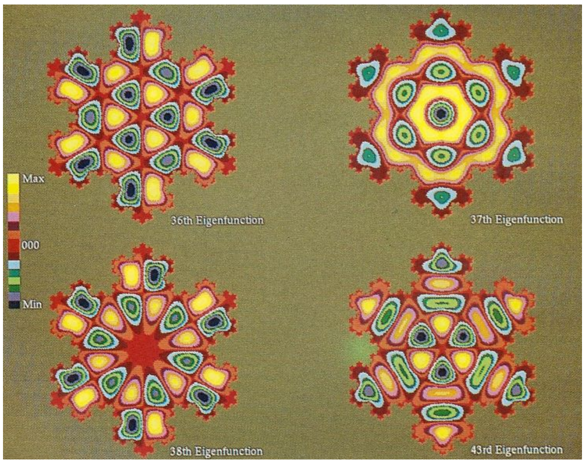


Figure 2: The eigenfunctions of the Dirichlet Laplacian can be considered as a mathematical model for the study of the (steady-states) "vibrations" of a drum. Here, they correspond to the fundamental tone and the overtones of a drum with very irregular ("fractal") boundary. In the graphics above, the drum's boundary is the Koch Snowflake curve [19].

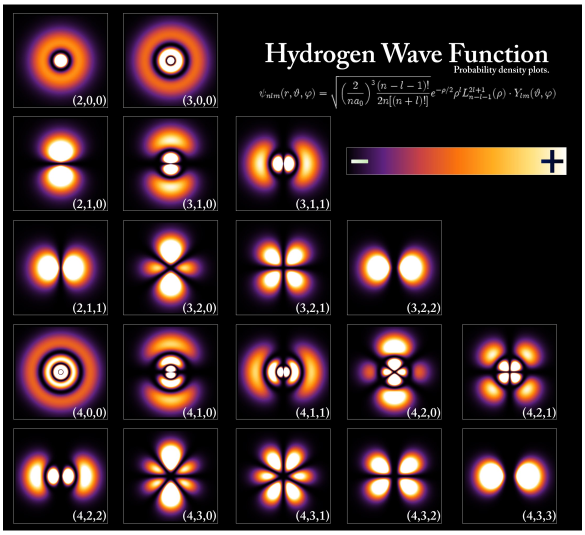


Figure 3: The electron probability density for the first few hydrogen atom electron orbitals shown as cross-sections. These orbitals form an orthonormal basis for the wave function of the electron. Different orbitals are depicted with different scale [30].

2 Hyperbolic Geometry

Hyperbolic geometry is a non-euclidean geometry obtained by Bolyai, Lobachevsky and Gauss independently. The difference between Euclidean geometry rise in the parallel postulate. In this chapter we follow [5], [2] for the two-dimensional case and [7] for the three-dimensional case.

2.1 Metric, measure and the Laplace operator

First of all, we have a Riemannian variety with an associated metric, a measure and the Laplace-Beltrami operator (we shall refer to such operator as the laplacian). The laplacian is defined over the metric in [5] and can be computed in local coordinates

$$\Delta = \sum_{i,j=1}^{n} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i} \left(\sqrt{g} g^{i,j} \frac{\partial}{\partial x_j} \right).$$

Where $ds^2 = \sum_{i,j=1}^{n} g_{i,j} dx_i dx_j$ is the metric written as a line element, $dv = \sqrt{g} \prod_{i=1}^{n} dx_i$ is the measure and $g = |\det((g_{i,j})_{i,j=1}^{n})|$ with $g^{i,j} := (((g_{i,j})_{i,j=1}^{n})^{-1})_{i,j}$ and $(g_{i,j})_{i,j=1}^{n} \in \mathbf{GL}_n(\mathbb{R})$.

So we start with $\mathbb{R}^3$ with the euclidian metric $ds^2 = {dx_1}^2 + {dx_2}^2 + {dx_3}^2$, the measure $dv = dx_1 dx_2 dx_3$ and therefore the laplacian $\Delta = \frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2} + \frac{\partial^2}{\partial {x_3}^2}$. We consider the following Riemann surface in our space, the hyperboloid $\mathbb{H}^2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : {x_3}^2 - {x_2}^2 - {x_1}^2 = 1\}$ with metric $ds^2 = {dx_1}^2 + {dx_2}^2 - {dx_3}^2$ (a non-definite positive symmetric two-tensor in $\mathbb{R}^3$ but definite positive in $\mathbb{H}^2$). We call this model the hyperboloid model. We apply the same process as above (metric, measure and laplacian) for three models of the upper sheet of the hyperboloid.

The first model is given by the polar coordinates (looking at our variety as a revolution surface) induced by the following map $p : U_j \times \mathbb{R} \to \mathbb{H}^2_+$, with $U_1 = (-\pi, \pi),\ U_2 = (0, 2\pi)$

$$p(\phi, \theta) = (\cos\phi \sinh\theta, \sin\phi \sinh\theta, \cosh\theta)$$

the induced metric, volume and laplace operator are

$$ds^2 = (\sinh\theta)^2 d\phi^2 + d\theta^2,$$

$$dv = \sinh\theta \; d\phi d\theta,$$

$$\Delta = \frac{1}{(\sinh\theta)^2} \frac{\partial^2}{\partial \phi^2} + \frac{1}{\sinh\theta} \left(\frac{\partial}{\partial\theta} \sinh\theta \frac{\partial}{\partial\theta} \right)$$

respectively.

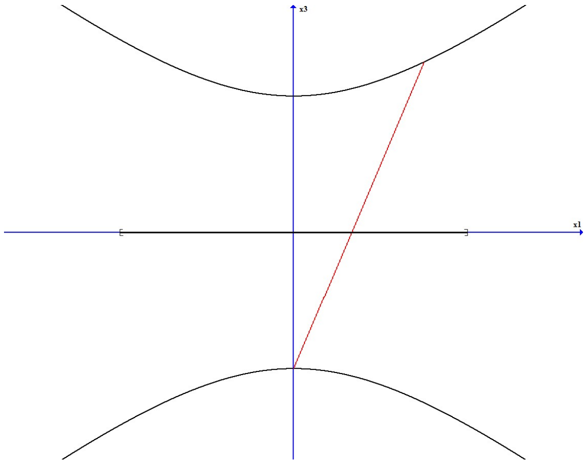


Figure 4: Hyperbolic stereographic projection

The second one, is the stereographic projection of the upper sheet $\pi : \mathbb{H}^2_+ \longrightarrow \mathbb{D}$ to the unit disk. So we have

$$\pi(x_1, x_2, x_3) = \left(\frac{x_1}{1+x_3}, \frac{x_2}{1+x_3}\right) = \frac{x_1 + x_2 i}{1+x_3} =: u_1 + u_2 i =: z$$

where $\mathbb{H}^2_+ := \{(x_1, x_2, x_3) \in \mathbb{H}^2 : x_3 > 0\}$ and $\mathbb{D} := \{z \in \mathbb{C} : |z| = 1\}$. The disk $\mathbb{D}$ with the induced metric by π is called the Poincaré model. A simple calculation gives the inverse function:

$$\pi^{-1}(z) = \left(\frac{2z}{1-|z|^2}, \frac{1+|z|^2}{1-|z|^2}\right) = \left(\frac{2u_1}{1-{u_1}^2-{u_2}^2}, \frac{2u_2}{1-{u_1}^2-{u_2}^2}, \frac{1+{u_1}^2+{u_2}^2}{1-{u_1}^2-{u_2}^2}\right).$$

When we project the upper sheet to the disk we modify the metric (and so the distances), the measure and the laplacian:

$$ds^2 = \frac{4dzd\bar{z}}{\left(1-|z|^2\right)^2} = \frac{4\left({du_1}^2 + {du_2}^2\right)}{\left(1-{u_1}^2-{u_2}^2\right)^2},$$

$$dv = \frac{4du_1du_2}{\left(1-{u_1}^2-{u_2}^2\right)^2},$$

$$\varDelta = \left(1-|z|^2\right)^2 \frac{\partial^2}{\partial z \partial \bar{z}} = \frac{\left(1-{u_1}^2-{u_2}^2\right)^2}{4}\left(\frac{\partial^2}{\partial {u_1}^2} + \frac{\partial^2}{\partial {u_2}^2}\right)$$

where $dz = du_1 + idu_2$, $d\bar{z} = \overline{dz} = du_1 - idu_2$, $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u_1} - i\frac{\partial}{\partial u_2}\right)$ and $\frac{\partial}{\partial \bar{z}} = \overline{\frac{\partial}{\partial z}} = \frac{1}{2}\left(\frac{\partial}{\partial u_1} + i\frac{\partial}{\partial u_2}\right)$.

We can observe that by the definition of the metric on the image space our projection becomes isometric.

We consider another model, the hyperbolic half-plane $\mathbb{H} := \{z \in \mathbb{C} : \Im z > 0\}$. We can obtain this model by a Möbius transformation of the Poincaré disk as follows:

$$\varphi : \mathbb{D} \longrightarrow \mathbb{H}, \quad \varphi(z) = i\frac{1+z}{1-z} =: x + iy =: w, \quad \varphi^{-1}(w) = \frac{w-i}{w+i}.$$

They induce the following metric, measure and laplacian:

$$ds^2 = -\frac{4dwd\bar{w}}{(w-\bar{w})^2} = \frac{dx^2+dy^2}{y^2},$$

$$dv = \frac{dxdy}{y^2},$$

$$\varDelta = (w-\bar{w})^2\frac{\partial^2}{\partial w\partial\bar{w}} = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

2.2 Isometries

We are interested in the transformations that leave our models invariant so, let's look first at the transformations that leave $\mathbb{H}^2$ invariant.

Definition 2.1. An isometry of Riemannian varieties of dimension n $(\Omega, (g_{i,j})_{i,j=1}^n)$, $(\Upsilon, (h_{i,j})_{i,j=1}^n)$ is a difeomorphism $f : \Omega \longrightarrow \Upsilon$ such that $f^*h = g$, where f^* is the pull-back of f [3].

The set of all isometries of a Riemannian variety $(\Omega, (g_{i,j})_{i,j=1}^n)$ is also a group $\mathrm{Isom}(\Omega) := \{f : \Omega \longrightarrow \Omega |\ f\ \textit{isometry}\}$. Let's construct $\mathrm{Isom}(\mathbb{H}^2)$. We start with the following bilinear symmetric form in $\mathbb{R}^3$,

$$h(x,y) = x_3y_3 - x_2y_2 - x_1y_1$$

Where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ are elements of $\mathbb{R}^3$. We begin with the transformations that leave h invariant. Let $\mathbb{R}^{3\times3}$ be the set of all 3×3 matrices with real coefficients. Any matrix M has three columns m_1, m_2, m_3, and we write $M = (m_1, m_2, m_3)$. We refer to the identity matrix as (e_1, e_2, e_3), where e_1, e_2, e_3 is the canonical basis of $\mathbb{R}^3$. Then we define the orthogonal group of h:

$$O_3(\mathbb{R}) := \{(m_1, m_2, m_3) \in \mathbb{R}^{3\times3} : h(m_i, m_j) = h(e_i, e_j),\ i, j = 1, 2, 3\}.$$

We have $Me_i = m_i,\ i = 1, 2, 3$ then:

$$M \in O_3(\mathbb{R}) \Longleftrightarrow h(Mx, My) = h(x, y)\ \ \forall x, y \in \mathbb{R}^3.$$

A simple calculation shows that this set is also a group. Let's look at the following matrices:

$$S_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad N_t = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix}, \quad \alpha, t \in \mathbb{R}.$$

Then it's easy to show that these transformations generate all the orthogonal group and that this group acts in $\mathbb{H}^2$ by isometries. We call Isom($\mathbb{H}^2$)= $O_3(\mathbb{R})$ the group of all isometries of the hyperboloid. Note that $S_1 = S_2 R_\pi$ and that S_3 represents in Isom($\mathbb{H}^2$) the map that interchange the two sheets of the hyperboloid.

Remark 2.1. *Note that* $N_t N_r = N_{t+r}, \quad R_\alpha R_\beta = R_{\alpha+\beta} \quad \forall t, r, \alpha, \beta \in \mathbb{R}$.

$$\mathrm{Isom}(\mathbb{H}^2) = \langle R_\alpha, N_t, S_2, S_3 \rangle_{\alpha, t \in \mathbb{R}}, \quad \mathrm{Isom}(\mathbb{H}^2_+) = \langle R_\alpha, N_t, S_2 \rangle_{\alpha, t \in \mathbb{R}}.$$

By our definition of the isometry group we obtain

$$\mathrm{Isom}(\mathbb{D}) = \pi \mathrm{Isom}(\mathbb{H}^2_+) \pi^{-1}, \quad \mathrm{Isom}(\mathbb{H}) = \varphi \mathrm{Isom}(\mathbb{D}) \varphi^{-1}.$$

Therefore, conjugating the generators S_2, R_α and N_t (for $\alpha, t \in \mathbb{R}$) via the isometries π, φ we obtain the generators of Isom($\mathbb{D}$) and Isom($\mathbb{H}$) respectively. If $z = x + yi$:

$$\begin{aligned} \tilde{\sigma}(z) := \pi\left(S_2\left(\pi^{-1}(z)\right)\right) &= \pi\left(S_2\left(\frac{2x}{1-x^2-y^2}, \frac{2y}{1-x^2-y^2}, \frac{1+x^2+y^2}{1-x^2-y^2}\right)\right) \\ &= \pi\left(\frac{2x}{1-x^2-y^2}, \frac{-2y}{1-x^2-y^2}, \frac{1+x^2+y^2}{1-x^2-y^2}\right) = \bar{z}, \end{aligned}$$

$$\sigma(z) := \varphi\left(\tilde{\sigma}\left(\varphi^{-1}(z)\right)\right) = \varphi\left(\overline{\frac{z-i}{z+i}}\right) = \varphi\left(\frac{\bar{z}+i}{\bar{z}-i}\right) = -\bar{z},$$

$$\begin{aligned} \tilde{\rho}_\alpha(z) := \pi\left(R_\alpha\left(\pi^{-1}(z)\right)\right) &= \pi\left(R_\alpha\left(\frac{2x}{1-x^2-y^2}, \frac{2y}{1-x^2-y^2}, \frac{1+x^2+y^2}{1-x^2-y^2}\right)\right) \\ &= \pi\left(\frac{2x\cos\alpha - 2y\sin\alpha}{1-x^2-y^2}, \frac{2x\sin\alpha + 2y\cos\alpha}{1-x^2-y^2}, \frac{1+x^2+y^2}{1-x^2-y^2}\right) = e^{i\alpha} z, \end{aligned}$$

$$\rho_\alpha(z) := \varphi\left(\tilde{\rho}_\alpha\left(\varphi^{-1}(z)\right)\right) = \varphi\left(e^{i\alpha}\frac{z-i}{z+i}\right) = \varphi\left(\frac{e^{i\alpha/2}z - ie^{i\alpha/2}}{e^{-i\alpha/2}z + ie^{-i\alpha/2}}\right)$$

$$= i\frac{z\left(e^{i\alpha/2}+e^{-i\alpha/2}\right)+\frac{\left(e^{i\alpha/2}-e^{-i\alpha/2}\right)}{i}}{i\left(e^{i\alpha/2}+e^{-i\alpha/2}\right)-z(e^{i\alpha/2}-e^{-i\alpha/2})}=\frac{z\cos(\alpha/2)-\sin(\alpha/2)}{\cos(\alpha/2)-z\sin(\alpha/2)},$$

$$\tilde{\nu}_t(z):=\pi\left(N_t\left(\pi^{-1}(z)\right)\right)=\pi\left(N_t\left(\frac{2x}{1-x^2-y^2},\frac{2y}{1-x^2-y^2},\frac{1+x^2+y^2}{1-x^2-y^2}\right)\right)$$

$$=\pi\left(\frac{2x\cosh t+(1+x^2+y^2)\sinh t}{1-x^2-y^2},\frac{2y}{1-x^2-y^2},\frac{(1+x^2+y^2)\cosh t+2x\sinh t}{1-x^2-y^2}\right)$$

$$=\frac{(1+x^2+y^2)\sinh t+2x\cosh t+2yi}{1-x^2-y^2+(1+x^2+y^2)\cosh t+2x\sinh t}$$

$$=\frac{(1+\cosh t)(1+x^2+y^2)\sinh t+x(1+(\sinh t)^2+2\cosh t+(\cosh t)^2)+2yi(1+\cosh t)}{(1+\cosh t)^2+2x\sinh t(1+\cosh t)+x^2(\sinh t)^2+y^2+(\sinh t)^2}$$

$$=\frac{(1+\cosh t)(1+x^2+y^2)\sinh t+(1+\cosh t)^2z+(\sinh t)^2\bar{z}}{(1+\cosh t+x\sinh t)^2+(\sinh t)^2y^2}$$

$$=\frac{((1+\cosh t)z+\sinh t)((\sinh t)\bar{z}+1+\cosh t)}{|z\sinh t+1+\cosh t|^2}=\frac{(1+\cosh t)z+\sinh t}{z\sinh t+1+\cosh t}$$

$$=\frac{z+\tanh\frac{t}{2}}{1+z\tanh\frac{t}{2}}=\frac{(\cosh t/2)z+\sinh t/2}{z\sinh t/2+\cosh t/2},$$

$$\nu_t(z):=\varphi\left(\tilde{\nu}_t\left(\varphi^{-1}(z)\right)\right)=\varphi\left(\frac{z\left(1+\tanh(\frac{t}{2})\right)-i\left(1-\tanh(\frac{t}{2})\right)}{z\left(1+\tanh(\frac{t}{2})\right)+i\left(1-\tanh(\frac{t}{2})\right)}\right)=\varphi\left(\frac{e^tz-i}{e^tz+i}\right)$$

$$=\varphi\left(\varphi^{-1}(e^tz)\right)=e^tz.$$

$$\Rightarrow \mathrm{Isom}(\mathbb{D})=\langle\tilde{\sigma},\tilde{\rho}_\alpha,\tilde{\nu}_t\rangle_{\alpha,t\in\mathbb{R}},\quad \mathrm{Isom}(\mathbb{H})=\langle\sigma,\rho_\alpha,\nu_t\rangle_{\alpha,t\in\mathbb{R}}.$$

2.3 Möbius Transformations

In this section we will focus on transformations such that leave $\hat{\mathbb{C}}:=\mathbb{C}\cup\{\infty\}$ invariant, for more details see [32]. Then it's not difficult to show that such transformations must have no essential singularities and other singularities different from poles and they must be rational functions $f:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$, $g:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$

$$f(z)=\frac{az+b}{cz+d},\ g(z)=\frac{a\bar{z}+b}{c\bar{z}+d},\quad a,b,c,d\in\mathbb{C}:\ ad-bc\neq 0,\ z\in\hat{\mathbb{C}}.$$

This functions are called Möbius transformations. We will associate a matrix, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in$ $\mathbf{GL}_2(\mathbb{C})$ to a transformation of the first type (f) if the determinant of the matrix is positive and to a transformation of the second type (g) if the determinant of the matrix is negative. One can check that this association is well defined and which gives a composition law to our transformations which implies that our set of transformations is also a group. That is, if we take two transformations f and g and make the composition fg, then the matrix of it's composition is FG where F and G are the matrix of f and g respectively, the identity map is associated to the identity matrix $\mathbb{1}$ and the inverse of a map is associated to the inverse of the associated matrix. So the association we have done it is also a morphism between the transformations which preserve the orientation (the first type) and the automorphisms of $\mathbb{P}^1_{\mathbb{C}}$ (called homographies) $\mathbf{PSL}_2(\mathbb{C}) = \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$ whith kernel

$$\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \in \mathbb{C} \setminus \{0\} \right\}.$$

Then we obtain that our group (denoted $\mathrm{Isom}^+(\hat{\mathbb{C}})$) is isomorphic to $\mathbf{PSL}_2(\mathbb{C})$. The projective class of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}_2(\mathbb{C})$ will be denoted by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{PSL}_2(\mathbb{C})$. In all this chapter we will write ∞ instead of $[1,0] \in \mathbb{P}^1_{\mathbb{C}}$ and $z \in \mathbb{C}$ instead of $[z,1] \in \mathbb{P}^1_{\mathbb{C}}$.

Definition 2.2. The cross-ratio between four points $z_0, z_1, z_2, z_3 \in \hat{\mathbb{C}}$ is

$$(z_0, z_1, z_2, z_3) = \frac{z_0 - z_2}{z_1 - z_2} : \frac{z_0 - z_3}{z_1 - z_3}$$

We will use the notation $(z_0, z_1, z_2) := (z_0, z_1, z_2, \infty) = \frac{z_0 - z_2}{z_1 - z_2}$.

Lemma 2.1. *The Möbius transformations preserve the cross-ratio.*

Proof. Take a Möbius transformation $f(z) = \frac{az+b}{cz+d}$ then a simple calculation shows that

$$(f(z_0), f(z_1), f(z_2)) = \frac{z_0 - z_2}{z_1 - z_2} \cdot \frac{cz_1 + d}{cz_0 + d} = (z_0, z_1, z_2)\frac{cz_1 + d}{cz_0 + d}$$

$$\Rightarrow (f(z_0), f(z_1), f(z_2), f(z_3)) = \frac{(f(z_0), f(z_1), f(z_2))}{(f(z_0), f(z_1), f(z_3))} = \frac{(z_0, z_1, z_2)}{(z_0, z_1, z_3)} = (z_0, z_1, z_2, z_3).$$

□

Lemma 2.2. *Three points are collinear (belong to the same line) if and only if its cross-ratio with infinity is a real number.*

Proof. Take $z, z_1, z_2 \in \hat{\mathbb{C}}$ then

$$\frac{z - z_2}{z_1 - z_2} = (z, z_1, z_2) = \overline{(z, z_1, z_2)} = \frac{\overline{z - z_2}}{\overline{z_1 - z_2}}$$

$$\iff (\overline{z_1} - \overline{z_2})(z - z_2) = (\overline{z} - \overline{z_2})(z_1 - z_2)$$
$$\iff 0 = (\overline{z_1} - \overline{z_2})z - (z_1 - z_2)\overline{z} + (z_1\overline{z_2} - \overline{z_1}z_2) =: Bz - \overline{Bz} + C$$

where C is imaginary pure. It's not difficult to see that any line is of the form $Bz - \overline{Bz} + C$ and that any curve of the form $Bz - \overline{Bz} + C$ is a line. □

Lemma 2.3. *Four points belong to the same circle if and only if its cross-ratio is a real number.*

Proof. Take $z, z_1, z_2 \in \hat{\mathbb{C}}$ then

$$\frac{z - z_2}{z_1 - z_2} : \frac{z - z_3}{z_1 - z_3} = (z, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)} = \frac{\overline{z - z_2}}{\overline{z_1 - z_2}} : \frac{\overline{z - z_3}}{\overline{z_1 - z_3}}$$
$$\iff (\overline{z_1} - \overline{z_2})(\overline{z} - \overline{z_3})(z - z_2)(z_1 - z_3) = (\overline{z} - \overline{z_2})(\overline{z_1} - \overline{z_3})(z_1 - z_2)(z - z_3)$$
$$\iff Az\overline{z} + Bz - \overline{Bz} + C = 0$$

where

$$A = (z_1 - z_3)(\overline{z_1} - \overline{z_2}) - (\overline{z_1} - \overline{z_3})(z_1 - z_2)$$
$$B = -\overline{z_3}(z_1 - z_3)(\overline{z_1} - \overline{z_2}) + \overline{z_2}(\overline{z_1} - \overline{z_3})(z_1 - z_2)$$
$$C = z_2\overline{z_3}(z_1 - z_3)(\overline{z_1} - \overline{z_2}) - z_3\overline{z_2}(\overline{z_1} - \overline{z_3})(z_1 - z_2)$$

where A and C are imaginary pure. The equation $Az\overline{z} + Bz - \overline{Bz} + C = 0$ belongs to a circle and in the case $A = 0$ belongs to a line (a circle containing infinity). We say that C is a generalized circle if its a circle or a line, i.e. if its given by an equation of a circle or a line respectively. □

Then, since Möbius transformations leave the corss-ratio invariant:

Corollary 2.3.1. *Möbius transformations map generalized circles to generalized circles.*

Proposition 2.1. *Let* $M = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} \in \boldsymbol{PSU}_{1,1}(\mathbb{C})$*. Then the following map gives an isomorphism between* $\boldsymbol{PSU}_{1,1}(\mathbb{C})$ *and* Isom$(\mathbb{D})$*:*

$$z \mapsto M[[z]] = \begin{cases} \frac{az+b}{\bar{b}z+\bar{a}}, & if \ \det M > 0; \\ \\ \frac{a+b\bar{z}}{\bar{b}+\bar{a}\bar{z}}, & if \ \det M < 0; \end{cases}$$

$z \in \mathbb{D}$.

Proof. In general we have that the group $\mathbf{GL}_2(\mathbb{C})$ acts by Möbius transformations in $\hat{\mathbb{C}}$. But only a subgroup of this one acts by isometries in $\mathbb{D}$. Take $f(z) = \frac{az+b}{cz+d} = M[[z]]$ with $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}_2(\mathbb{C})$ then this transformation still not change the metric:

$$ds^2 = \frac{4df(z)d\bar{f}(z)}{\left(1 - |f(z)|^2\right)^2} = \frac{4\frac{\partial f(z)}{\partial z}\frac{\partial f(\bar{z})}{\partial \bar{z}}dzd\bar{z}}{\left(1 - |f(z)|^2\right)^2}$$

$$= \frac{4dzd\bar{z}(\det M)^2}{\left(\left(|c|^2 - |a|^2\right)|w|^2 + (c\bar{d} - a\bar{b})w + (\bar{c}d - \bar{a}b)\bar{w} + |d|^2 - |b|^2\right)^2}$$

$$= \frac{4dzd\bar{z}}{\left(\frac{|d|^2-|b|^2}{\det M} + \frac{c\bar{d}-a\bar{b}}{\det M}w + \frac{\bar{c}d-\bar{a}b}{\det M}\bar{w} - \frac{|a|^2-|c|^2}{\det M}|w|^2\right)^2}.$$

Then we have:

$$\det M = |d|^2 - |b|^2 = |a|^2 - |c|^2, \quad \bar{c}d - \bar{a}b = c\bar{d} - a\bar{b} = 0$$

$$\Rightarrow ad = \det M + bc \Longrightarrow c|d|^2 = ad\bar{b} = \bar{b}\det M + |b|^2 c$$

$$c\det M = c\left(|d|^2 - |b|^2\right) = \bar{b}\det M \Longrightarrow c = \bar{b}, \quad d = \bar{a}.$$

Let $\alpha = \arg a$ and $\beta = \arg b$. Then we have the following formulas for $f(z)$:

$$f(z) = \tilde{\rho}_{\alpha+\beta}\left(\tilde{\nu}_{2\operatorname{arctgh}\frac{|b|}{|a|}}\left(\tilde{\rho}_{\alpha-\beta}(z)\right)\right)$$

if $\det M > 0$ ($|a| > |b|$),

$$f(z) = \tilde{\rho}_{\alpha+\beta}\left(\tilde{\nu}_{2\operatorname{arctgh}\frac{|a|}{|b|}}\left(\tilde{\rho}_{\beta-\alpha}\left(\tilde{\sigma}(z)\right)\right)\right) = \tilde{\rho}_{\alpha+\beta}\left(\tilde{\nu}_{2\operatorname{arctgh}\frac{|a|}{|b|}}\left(\tilde{\sigma}\left(\tilde{\rho}_{\alpha-\beta}(z)\right)\right)\right)$$

if $\det M < 0$ ($|a| < |b|$).
It's easy to check that the mentioned map respects the group laws between the two groups and that it is also an isomorphism. □

Proposition 2.2. *Let* $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \boldsymbol{PSL}_2(\mathbb{R})$. *Then the following map gives an isomorphism between* $\boldsymbol{PSL}_2(\mathbb{R})$ *and* Isom$(\mathbb{H})$*:*

$$z \mapsto M[z] = \begin{cases} \frac{az+b}{cz+d}, & if \ \det M > 0; \\ \\ \frac{a\bar{z}+b}{c\bar{z}+d}, & if \ \det M < 0; \end{cases}$$

$z \in \mathbb{H}$.

Proof. We repeat the same argument as before so, let's take $f(z) = \frac{az+b}{cz+d}$ with $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}_2(\mathbb{C})$ then this transformation still not change the metric:

$$ds^2 = -\frac{4df(z)d\bar{f}(z)}{\left(f(z) - \bar{f}(z)\right)^2} = -\frac{4\frac{\partial f(z)}{\partial z}\frac{\partial f(\bar{z})}{\partial \bar{z}}dzd\bar{z}}{\left(f(z) - \bar{f}(z)\right)^2}$$

$$= -\frac{4dzd\bar{z}(\det M)^2}{\left((a\bar{c} - \bar{a}c)|z|^2 + (a\bar{d} - \bar{b}c)z - (\bar{a}d - b\bar{c})\bar{z} + b\bar{d} - \bar{b}d\right)^2}$$

$$= -\frac{4dzd\bar{z}}{\left(\frac{a\bar{c}-\bar{a}c}{\det M}|z|^2 + \frac{a\bar{d}-\bar{b}c}{\det M}z - \frac{\bar{a}d-b\bar{c}}{\det M}\bar{z} + \frac{b\bar{d}-\bar{b}d}{\det M}\right)^2}.$$

Then without loss of generality we can fix:

$$\det M = a\bar{d} - \bar{b}c = \bar{a}d - b\bar{c}, \quad a\bar{c} - \bar{a}c = b\bar{d} - \bar{b}d = 0$$

$$\Rightarrow ad = \det M + bc, \;\; a\bar{d} = \det M + \bar{b}c$$

$$\begin{cases} a\bar{c}d = \det M\bar{c} + b|c|^2 \\ \bar{a}cd = \det Mc + b|c|^2 \end{cases} \Rightarrow c = \bar{c} \Rightarrow c \in \mathbb{R}.$$

By analogy, if we repeat the same process we obtain $a, b, c, d \in \mathbb{R}$. Of course this map (as in the other case) is an isomoprhism. □

2.4 Distances and Geodesics

Geodesics are curves that minimize the distance between any two points of the curve. With this definition one can find the Euler-Lagrange equations of the geodesics of a manifold (as a variational problem). In this section we will use another well known definition which of course coincides with the mentioned above.

Let ∇ be the Levi-Civita connection of our Riemannian manifold M of dimension m with local coordinates $(x_i)_{i=1}^m$. Then, a curve $\gamma : I \subset \mathbb{R} \to M$ $\gamma(t) = (\gamma_k(t))_{k=1}^m$ is a geodesic if, and only if $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$. If Γ_{ij}^k are the Christoffel symbols of the connection defined as

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^m \Gamma_{ij}^k \frac{\partial}{\partial x_k}$$

where $\left\{\frac{\partial}{\partial x_i}\right\}_{i=1}^m$ is the induced (by the chart) local basis of the tangent vector bundle TM then the above definition of the geodesics reads:

$$\ddot{\gamma}_k + \sum_{i,j=1}^m \Gamma_{ij}^k \dot{\gamma}_i \dot{\gamma}_j = 0 \quad k = 1, ..., m.$$

There's a way to calculate the Christoffel symbols in local coordinates of the Levi-Civita connection with metric $(g_{ij})_{i,j}$ by the formula

$$\Gamma_{ij}^k = \sum_{r=1}^m \frac{1}{2} g^{kr} \left(\frac{\partial g_{jr}}{\partial x_i} + \frac{\partial g_{ir}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_r} \right).$$

Since we established the basic definitions we proceed to do some calculations. We start with the polar coordinates model for the hyperboloid. The Christoffel symbols of the associated Levi-Civita connection are

$$\begin{array}{ll} \Gamma_{\phi\phi}^{\phi} = 0, & \Gamma_{\phi\theta}^{\phi} = \Gamma_{\theta\phi}^{\phi} = \coth\theta, \\ \Gamma_{\theta\theta}^{\theta} = 0, & \Gamma_{\phi\theta}^{\theta} = \Gamma_{\theta\phi}^{\theta} = 0. \end{array}$$

Where we use the variable symbols indexing instead of the numerical one. Then we obtain the geodesic equations

$$\ddot{\phi} + 2\coth\theta\ \dot{\phi} = 0, \quad \ddot{\theta} = 0.$$

Then we have that the coordinate curves $\phi = \text{const},\ \theta = t$ with $t \in \mathbb{R}$ are geodesics. These geodesics are straight lines which they intersect in the origin when they are projected to the disk and since the Möbius transformations preserve angles and generalized circles we deduce that all generalized circles which intersect the boundary of the disk orthogonally are geodesics (two curves intersect orthogonally if they tangents do). We also have that all geodesics are generalized circles which intersect orthogonally the boundary of the disk because the Möbius transformations are determined by three points (and generalized circles are determined by three points) and then, given two different, arbitrary points u, v we can map it to the points -1 and 1 respectively and then we obtain that the geodesic such that connects u with v is a generalized circle which intersects orthogonally to the boundary of the disk.

In the case of the hyperbolic half plane we have that the geodesics are also the generalized circles which intersect orthogonally to the boundary $\partial\mathbb{H} = \mathrm{P}^1_{\mathrm{C}}$ because the isometry which maps $\mathbb{D}$ to $\mathbb{H}$ is a Möbius transformation.

Take two points $x, y \in \Omega$ (with $\Omega \in \{\mathbb{H}^2_+, \mathbb{D}, \mathbb{H}\}$) and $\gamma(t)$ the geodesic with $\gamma(0) = x$ and $\gamma(t_0) = y$, $t_0 \in \mathbb{R}$ (there is a unique solution for the differential equation, with t_0 the minimum such that $\gamma(t_0) = y$), then the distance between x and y is denoted by $d(x, y)$ and it is defined as

$$d_\Omega(x, y) = \int_{\gamma(0)}^{\gamma(t_0)} ds_\Omega = \int_0^{t_0} \sqrt{g_\Omega(\dot{\gamma}, \dot{\gamma})}\ dt,$$

where we remark that the distance, the metric and the norm of a vector depends on the model Ω. Since we know that isometries preserve the metric and therefore the distances we have that any geodesic mapped by an isometry is a geodesic, i.e. $\mathrm{Isom}(\Omega)\gamma(t)$ is a set of geodesics and in our case it's obvious that is the set of all geodesics (take any geodesic and construct the isometry which maps the geodesic to a straight line in the disk model, it is sufficient to do it in one model).

Take the geodesic $\gamma(t) = (x_1, 0, x_3) := (\sinh t, 0, \cosh t) = N_t(0, 0, 1)$ then it's length from $t = 0$ to $t = \theta = \operatorname{arcosh} x_3$ can be easily computed (using that $\|\dot{\gamma}\|^2 = g_{\mathbb{H}^2}(\dot{\gamma}, \dot{\gamma}) = 1$)

$$d((0,0,1), (x_1, 0, x_3)) = d(\gamma(0), \gamma(\theta)) = \int_{\gamma(0)}^{\gamma(\theta)} ds = \int_0^\theta dt = \theta.$$

We pretend to find a formula for the distance in the given models and the model of polar coordinates gives the formula in a suitable way. Taking polar coordinates in the Poincaré disk

$$z = u_1 + iu_2 = \frac{x_1 + ix_2}{1 + x_3} = \mathrm{e}^{i\phi} \tanh\frac{\theta}{2} =: \rho \mathrm{e}^{i\phi},$$

$$\theta = \operatorname{arcosh} x_3 = \operatorname{arcosh} \frac{1+\rho^2}{1-\rho^2} = \log \frac{1+\rho}{1-\rho} = \log(0, \rho, 1, -1).$$

To find the general formula for distance between to arbitrary points on the Poincaré disk z and $z' = re^{i\alpha}$ we use isometries to map z' to 0 (without changing distances) and then apply the last formula.

$$f := \tilde{\nu}_{-2\operatorname{arctgh}(r)}\ \tilde{\rho}_{-\alpha} \Rightarrow f(z') = 0 \Rightarrow f(z) = \frac{z\mathrm{e}^{-i\alpha} - r}{1 - zr\mathrm{e}^{-i\alpha}} = \mathrm{e}^{-i\alpha} \frac{z - z'}{1 - z\overline{z'}}$$

$$\Rightarrow \cosh d(z, z') = \cosh d(f(z), 0) = \frac{1 + |f(z)|^2}{1 - |f(z)|^2} = \frac{|1 - z\overline{z'}|^2 + |z - z'|^2}{|1 - z\overline{z'}|^2 - |z - z'|^2}$$

$$\Rightarrow d(z, z') = \log \frac{|1 - z\overline{z'}| + |z - z'|}{|1 - z\overline{z'}| - |z - z'|} = 2\operatorname{arctanh} \left| \frac{z - \overline{z'}}{1 - z\overline{z'}} \right| = \log(z, z', z'_\infty, z_\infty),$$

where z_∞ and z'_∞ are the infinity points of z and z' respectively, by the extension of the geodesic which connects them. That is, let γ be the geodesic such that $\gamma(0) = z$ and $\gamma(1) = z'$ then $\gamma(\{t < 0\}) \cap \partial\mathbb{D} = \{z_\infty\}$ and $\gamma(\{t > 0\}) \cap \partial\mathbb{D} = \{z'_\infty\}$. To find the formula in the hyperbolic half plane we take two arbitrary points $w,\ w' \in \mathbb{H}$ where $z = \varphi^{-1}(w)$ and $z' = \varphi^{-1}(w')$ with $z, z' \in \mathbb{D}$. Then using that φ is an isometry and that Möbius transformations preserve the corss-ratio

$$z - z' = 2i\frac{w - w'}{(w+i)(w'+i)}, \quad 1 - z\overline{z'} = -2i\frac{w - \overline{w'}}{(w+i)(\overline{w'} - i)} \Rightarrow \left| \frac{z - \overline{z'}}{1 - z\overline{z'}} \right| = \left| \frac{w - w'}{w - \overline{w'}} \right|$$

$$\Rightarrow \cosh d_{\mathbb{H}}(w, w') = \cosh d_{\mathbb{D}}(z, z') = \frac{|w - \overline{w'}|^2 + |w - w'|^2}{|w - \overline{w'}|^2 - |w - w'|^2} = 1 + \frac{|w - w'|^2}{2\Im(w)\Im(w')}$$

$$\Rightarrow d(w, w') = \log \frac{|w - \overline{w'}| + |w - w'|}{|w - \overline{w'}| - |w - w'|} = 2\operatorname{arctanh} \frac{|w - w'|}{|w - \overline{w'}|} = \log(w, w', w'_\infty, w_\infty).$$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{PSL}_2(\mathbb{R})$ then

$$|i - A[i]|^2 = \left| i - \frac{ai + b}{ci + d} \right|^2 = \left| \frac{-c - b + i(d - a)}{ci + b} \right|^2 = \frac{a^2 + b^2 + c^2 + d^2 - 2}{c^2 + d^2}$$

$$\Longrightarrow\ 2\cosh d(i, A[i]) = 2 + \frac{|i - A[i]|^2}{\Im(i)\Im(A[i])} = a^2 + b^2 + c^2 + d^2.$$

Proposition 2.3. *Let $p, w \in \mathbb{D}$ be such that:*

$$w = \frac{2p}{1 + |p|^2}, \quad p = \frac{w}{1 + \sqrt{1 - |w|^2}}.$$

1) *The* ***half-turn*** *at center p is an isometry $m_p \in Isom(\mathbb{D})$ represented by the following matrix*

$$m_p = \frac{i}{\sqrt{1-|w|^2}} \begin{bmatrix} 1 & -w \\ \bar{w} & -1 \end{bmatrix} = \frac{i}{1-|p|^2} \begin{bmatrix} 1+|p|^2 & -2p \\ 2\bar{p} & -(1+|p|^2) \end{bmatrix}$$

This isometry satisfies:

$$m_p(z) = z \iff z = p, \quad m_p^2 = -\mathbb{1}.$$

2) *Let $z_1, z_2 \in \mathbb{D}$ such that p is the midpoint of the geodesic arc from z_1 to z_2 (so that $m_p(z_1) = z_2$ and $m_p(z_2) = z_1$). Then p and m_p are given in the above form with*

$$w = \frac{z_1(1-|z_2|^2) + z_2(1-|z_1|^2)}{1-|z_1 z_2|^2}.$$

3) *There exists a point $q \in \mathbb{C} \cup \{\infty\}$, $|q| > 1$ and a circle C in $\mathbb{C}$ with center q (in the Euclidean sense) ortogonal to the boundary $\partial\mathbb{D}$ of $\mathbb{D}$ defining a geodesic $\gamma = C \cap \mathbb{D}$. Then the* ***symmetry*** *of axis γ is an isometry $m_\gamma \in Isom(\mathbb{D})$ represented by the following matrix*

$$m_\gamma = \frac{i}{\sqrt{|q|^2 - 1}} \begin{bmatrix} 1 & -q \\ \bar{q} & -1 \end{bmatrix}$$

This isometry satisfies:

$$m_\gamma(z) = z \iff z \in \gamma, \quad m_\gamma^2 = \mathbb{1}.$$

Proof. 1) The main idea of the proof is to map p to the origin and then apply the half-turn $\tilde{\rho}_\pi$.

Let $p = re^{i\varphi}$ then we know that $\tilde{\rho}_{-\varphi}$ represents a rotation such that p is mapped into the geodesic line $\mathbb{D} \cap \mathbb{R}$. Then we can map p to the origin by the transformation $\tilde{\nu}_{-2\text{arctanh}(r)}$. It will be easier to calculate m_p if we use the matrix representation:

$$\tilde{\rho}_{-\varphi} = \begin{bmatrix} e^{-i\frac{\varphi}{2}} & 0 \\ 0 & e^{i\frac{\varphi}{2}} \end{bmatrix} \quad \tilde{\nu}_{-2\text{arctanh}(r)} = \begin{bmatrix} 1 & -r \\ -r & 1 \end{bmatrix}$$

We define:

$$g := \tilde{\nu}_{-2\text{arctanh}(r)} \tilde{\rho}_{-\varphi} = \frac{1}{\sqrt{1-r^2}} \begin{bmatrix} e^{-i\frac{\varphi}{2}} & -re^{i\frac{\varphi}{2}} \\ -re^{-i\frac{\varphi}{2}} & e^{i\frac{\varphi}{2}} \end{bmatrix}$$

We know that a half-turn from the origin can be represented by the matrix

$$f := \tilde{\rho}_\pi = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

Then by the mentioned process we have:

$$m_p = g^{-1} f g$$

The unique fixed point of m_p is $z = p$, since the unique fixed point of f is $z = 0$. We also have:

$$m_p^2 = g^{-1} f^2 g = -g^{-1} g = -\mathbb{1}$$

2)

$$z_2 = m_p(z_1) = \frac{z_1 - w}{\bar{w} z_1 - 1} \Rightarrow z_1 - w = z_2(\bar{w} z_1 - 1)$$

$$\Rightarrow w = z_1 - \bar{w} z_1 z_2 + z_2 \Rightarrow z_1 z_2 \bar{w} = z_2 |z_1|^2 - w|z_1 z_2|^2 + z_1 |z_2|^2$$

$$\Rightarrow w(1 - |z_1 z_2|^2) = z_1(1 - |z_2|^2) + z_2(1 - |z_1|^2)$$

$$\Rightarrow w = \frac{z_1(1 - |z_2|^2) + z_2(1 - |z_1|^2)}{1 - |z_1 z_2|^2}.$$

3) Take q the point in the (Euclidean) line, generated by 0 and p, such that lies in the intersection of the two tangent lines to $z_1, z_2 \in \partial\mathbb{D}$ respectively (suppose the three lines intersect in a point of $\mathbb{C} \cup \{\infty\}$). Since three complex points lie in the same line iff the cross-ratio of this points with the infinity is a real number:

$$\lambda := \frac{q}{w} = (q, w, 0) = \frac{1}{1 + |p|^2}(q, p, 0) \in \mathbb{R}$$

Construct a circle C of center q and of a radius such that $z_1, z_2 \in C$. Then we have by the Pythagoras theorem that the point q must satisfy the following equalities:

$$|q - z_1|^2 + 1 = |q|^2, \quad |q - z_2|^2 + 1 = |q|^2$$

$$\Rightarrow 2\Re(\bar{q} z_1) = q\bar{z_1} + \bar{q} z_1 = 2, \quad 2\Re(\bar{q} z_2) = q\bar{z_2} + \bar{q} z_2 = 2$$

$$\Rightarrow 2 = q\frac{\bar{z_1} - \bar{w}}{w\bar{z_1} - 1} + \bar{q}\frac{z_1 - w}{\bar{w} z_1 - 1} = \frac{q(\bar{z_1} - \bar{w})(\bar{w} z_1 - 1) + \bar{q}(z_1 - w)(w\bar{z_1} - 1)}{|\bar{w} z_1 - 1|^2}$$

$$= \frac{\overline{q(w|z_1|^2 - z_1 + w - w^2 \bar{z_1})} + \bar{q}(w|z_1|^2 - z_1 + w - w^2 \bar{z_1})}{|w z_1|^2 - 2\Re(\bar{w} z_1) + 1}$$

$$= \frac{4\Re(\bar{q} w) - 2\Re(\bar{q} z_1) - 2\Re(\bar{q} w^2 \bar{z_1})}{|w|^2 - 2\Re(\bar{q} z_1)/\lambda + 1} = \frac{2|w|^2(2\lambda - 1) - 2}{|w|^2 - 2/\lambda + 1}$$

$$\Rightarrow 2(|w|^2 - 2/\lambda + 1) = 2|w|^2(2\lambda - 1) - 2 \Rightarrow \left(|w|^2 - \frac{1}{\lambda}\right)(1 - \lambda) = 0$$

Note that $1 < |q| = |\lambda||w| < |\lambda|$. Then we finally have:

$$\lambda = \frac{1}{|w|^2} \Rightarrow q = \lambda w = \frac{w}{|w|^2} = \frac{1}{\bar{w}}$$

Since we obtained an analytic expression of q we've proved that the three mentioned lines intersect in this point (if we suppose $z_1 \neq z_2$) and it also exists a radius such that C intersect orthogonaly $\partial\mathbb{D}$ at z_1, z_2.

To find an expression of m_γ we repeat the same process as m_p with $p = re^{i\varphi}$ but now we are mapping the geodesic represented by p to the geodesic $\mathbb{D} \cap \mathbb{R}$ and then we apply the symmetry of axis $\mathbb{D} \cap \mathbb{R}$ $z \mapsto \bar{z}$:

$$g := \tilde{\rho}_{\pi/2}\tilde{\nu}_{-2\mathrm{arctanh}(r)}\tilde{\rho}_{-\varphi}, , \quad f := \tilde{\sigma} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$m_\gamma = g^{-1}fg = \frac{i}{\sqrt{|q|^2 - 1}} \begin{bmatrix} 1 & -q \\ \bar{q} & -1 \end{bmatrix}$$

The unique fixed points of m_γ are those such that $z \in \gamma$, since the unique fixed points of f are those such that $z \in \mathbb{D} \cap \mathbb{R}$. We also have:

$$m_\gamma^2 = g^{-1}f^2g = g^{-1}g = \mathbb{1}.$$

□

Proposition 2.4. *Let $p = r + si \in \mathbb{H}$, $r, s \in \mathbb{R}$, $s > 0$:*
1) *The* ***half-turn*** *at center p is an isometry $h_p \in Isom(\mathbb{H})$ represented by the following matrix*

$$h_p = \frac{1}{s} \begin{bmatrix} -r & r^2 + s^2 \\ -1 & r \end{bmatrix}$$

This isometry satisfies:

$$h_p(z) = z \iff z = p, \quad h_p^2 = \mathbb{1}.$$

2) *Let $r \in \mathbb{R}$ be the center and $s > 0$ the radius of a half circle representing a geodesic. Then the following matrix represents the* ***symmetry*** *$h_\gamma : \mathbb{H} \to \mathbb{H}$ with axis γ,*

$$h_\gamma = \frac{1}{s} \begin{bmatrix} r & -r^2 + s^2 \\ 1 & -r \end{bmatrix}$$

Proof. 1) It is not difficult to see that the isometry f sends p to i:

$$f = \frac{1}{\sqrt{s}} \begin{bmatrix} 1 & -r \\ 0 & s \end{bmatrix} \implies f^{-1} = \frac{1}{\sqrt{s}} \begin{bmatrix} s & r \\ 0 & 1 \end{bmatrix}.$$

Since $h_i = \rho_\pi$ is a half-turn at center i a simple calculation shows that

$$h_p = f^{-1}h_if = \frac{1}{s} \begin{bmatrix} -r & r^2 + s^2 \\ -1 & r \end{bmatrix}.$$

2) Take $p = r + is$ the midpoint of the geodesic line γ (since γ is a half-circle of center r and radius s). We proceed as above taking the conjugation $\rho_{-\pi/2}\sigma\rho_{\pi/2}$ instead of the rotation ρ_π (because $\rho_{-\pi/2}\sigma\rho_{\pi/2}$ is a symmetry with axis the vertical line connecting 0 with i). That is, we send the midpoint p to i and then we apply the symmetry of the corresponding axis

$$h_\gamma = f^{-1}\rho_{-\pi/2}\sigma\rho_{\pi/2}f = \frac{1}{s} \begin{bmatrix} r & -r^2 + s^2 \\ 1 & -r \end{bmatrix}.$$

□

2.5 Orientation preserving isometries

In this section we classify the isometries of the hyperbolic plane such that preserve the orientation (which means that we're not able to use complex conjugation) using the upper half-plane model. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{PSL}_2(\mathbb{R}),\ A \neq \mathbb{1}$ and suppose that $z \in \hat{\mathbb{C}}$ is fixed by the action of A. Then we have

$$z = A[z] = \frac{az+b}{cz+d} \Rightarrow cz^2 + (d-a)z - b = 0 \Rightarrow z = \frac{a - d \pm \sqrt{(\mathrm{tr}A)^2 - 4}}{2c}.$$

Then we have three cases (note that the trace of an element of $\mathbf{PSL}_2(\mathbb{C})$ is only well defined under change of sign and then $|\mathrm{tr}|$ is well defined in $\mathbf{PSL}_2(\mathbb{C})$):
– If $|\mathrm{tr}A| < 2$, then A has two fixed points p and $\overline{p}$ one of them (say p) is in $\mathbb{H}$. In this case, A is said to be of *elliptic type*.
– If $|\mathrm{tr}A| = 2$, then A has a unique fixed point $p \in \mathbb{R} \cup \{\infty\} = \partial\mathbb{H}$. In this case, A is said to be of *parabolic type*.
– If $|\mathrm{tr}A| > 2$, then A has two fixed points in $p \in \mathbb{R} \cup \{\infty\} = \partial\mathbb{H}$. In this case, A is said to be of *hyperbolic type*.
– If $\mathrm{tr}A \in \mathbb{C}$, then A has two complex fixed points. In this case, A is said to be of *loxodromic type*. This case doesn't occur in $\mathbb{H}$ because $\mathrm{tr}A \in \mathbb{R}$ always.

2.5.1 Elliptic Type

Suppose that the transformation A has $z = i$ as fixed point. Then we can take $d = a$ (since we have a free parameter imposing $\det A = 1$) so

$$i = \frac{\sqrt{a^2-1}}{c} \Rightarrow a^2 + c^2 = 1 \Rightarrow \exists\ \alpha \in]-\pi,\pi] :\ a = \cos\frac{\alpha}{2},\ c = -\sin\frac{\alpha}{2},\ b = -c.$$

Then we have shown that our matrix A is in the same class of the matrix of ρ_α. Remember that the representation matrix of the transformation ρ_α is

$$\rho_\alpha = \begin{bmatrix} \cos\frac{\alpha}{2} & \sin\frac{\alpha}{2} \\ -\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{bmatrix}.$$

The corresponding isometry of the unit disk $\tilde{\rho}_\alpha$ is a Euclidean rotation with angle of rotation α. Since the Euclidean and hyperbolic angle measure coincide and $\varphi : \mathbb{D} \to \mathbb{H}$ preserves angles it follows that if v is a unit tangent vector at point i and ρ_α sends v to v' then the angle between v and v' is α.

Let f be any elliptic transformation with fixed point $p = r + is$ then, if $\mu := \frac{1}{\sqrt{s}}\begin{bmatrix} 1 & -r \\ 0 & s \end{bmatrix}$ we have that $g = \mu f \mu^{-1}$ is an elliptic transformation with i as fixed point. So, we have that $g = \rho_\alpha$ for some $\alpha \in]-\pi,\pi]$, that is $f = \mu^{-1}\rho_\alpha\mu$ (f is in the conjugacy class of ρ_α, $\mathcal{C}(\rho_\alpha)$). We say that f is a *rotation*, p is its *center*, and α is its *angle of rotation.* Using that $\mathrm{tr}(\mu^{-1}\rho_\alpha\mu) = \mathrm{tr}\rho_\alpha$ (the trace is invariant under conjugation) we arrive at the following result: *Any elliptic transformation $f \in \mathbf{PSL}_2(\mathbb{R})$, is a rotation, and its angle of rotation α satifies*

$$2\cos(\alpha/2) = |\mathrm{tr} f|.$$

Note that the same holds in $\mathbb{D}$ with the corresponding isometry group.

Proposition 2.5. *The rotation f with center p and angle α is given as follows:*

a) If $p \in \mathbb{D}$

$$f = \cos(\alpha/2)\mathbb{1} + \sin(\alpha/2)m_p.$$

b) If $p \in \mathbb{H}$

$$f = \cos(\alpha/2)\mathbb{1} + \sin(\alpha/2)h_p.$$

Proof. Let g be an isometry which maps p to the origin (0 if $p \in \mathbb{D}$, i if $p \in \mathbb{H}$). If $p \in \mathbb{D}$ we have that gfg^{-1} is a rotation with center 0 and angle α, that is $gfg^{-1} = \tilde{\rho}_\alpha$. Then

$$gfg^{-1} = \tilde{\rho}_\alpha = \begin{bmatrix} \mathrm{e}^{i\alpha/2} & 0 \\ 0 & \mathrm{e}^{-i\alpha/2} \end{bmatrix} = \cos(\alpha/2)\ \mathbb{1} + \sin(\alpha/2)\ m_0$$

which shows our claim since $m_p = g^{-1}m_0 g$. A similar argument using h_p instead of m_p shows b) if $p \in \mathbb{H}$. □

2.5.2 Hyperbolic Type

Let A represent a Möbius transformation with fixed points 0 and ∞. This implies $b = 0$ and $c = 0$. So $d = a^{-1}$ since $a \neq 0$. There exists $t \in \mathbb{R}$ such that $a = e^{t/2}$. We obtain that A is in the same class of ν_t

$$\nu_t = \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}.$$

Now let $f \in \mathbf{PSL}_2(\mathbb{R})$ be any hyperbolic transformation with two diferent fixed points $u, w \in \partial\mathbb{H}$. Its axis is $\gamma_f = C \cap \mathbb{H}$ where C is a generalized circle interecting $\partial\mathbb{H}$ orthogonally at u and w (called the endpoints at infinity of γ_f). Then one can construct a transformation μ such that maps γ_f to the geodesic with 0 and ∞ as enpoints at infinity. Take $\mu = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{PSL}_2(\mathbb{R})$ with $d = 1$ (since we have a free parameter imposing $\det\mu = 1$), then one easily obtains

$$\mu = \begin{bmatrix} \frac{w}{w-u} & \frac{-uw}{w-u} \\ \frac{-1}{w} & 1 \end{bmatrix}.$$

Since $g = \mu f \mu^{-1}$ is a hyperbolic transformation which fixes 0 and ∞ we conclude that exists $t \in \mathbb{R}$ such that $g = \nu_t$ and then $f = \mu^{-1}\nu_t\mu$ (f is in the conjugacy class of ν_t,

$\mathcal{C}(\nu_t))$.

Take a point $p = r + is \in \mathbb{H}$ then by the distance formula we have

$$d(p, \nu_t(p)) = d(p, \mathrm{e}^t p) = 1 + \frac{|p|^2|1 - \mathrm{e}^t|^2}{2s^2\mathrm{e}^t} \geq 1 + \frac{(1 - \mathrm{e}^t)^2}{2} = \cosh t.$$

The equality yields if and only if $r = 0$, i.e. if and only if p lies in the axis of ν_t. Then we have that $\mathrm{tr}(f) = \mathrm{tr}(\nu_t) = 2\cosh(t/2)$. Observing that $p \in \mathbb{H}$ lies on γ_f if and only if $\mu(p)$ lies on the axis of ν_t, and using that

$$d(p, f(p)) = d(\mu(p), \mu(f(p))) = d(\mu(p), \nu_t(\mu(p))),$$

we conclude the following.

Theorem 2.4. *Let $f \in \mathbf{PSL}_2(\mathbb{R})$ be any hyperbolic transformation and ℓ_f such that*

$$2\cosh(\ell_f/2) = |\mathrm{tr} f|.$$

Then $d(p, f(p)) \geq \ell_f$, for any $p \in \mathbb{H}$, with equality if and only if p lies on the axis of f. $\square$

Then number ℓ_f is called the *displacement length* of f.

Proposition 2.6. *a) Let $q \in \mathbb{C}$, $|q| > 1$ be the center of the generalized circle representing a geodesic γ in $\mathbb{D}$. The hyperbolic isometry of axis γ and displacement length $\ell \geq 0$ is given by*

$$f = \cosh(\ell/2)\mathbb{1} + \sinh(\ell/2)m_\gamma.$$

b) Let $r \in \mathbb{R}$ be the center and $s > 0$ the radius of a generalized circle representing a geodesic $\gamma \in \mathbb{H}$. The hyperbolic isometry with axis γ and displacement length $\ell \geq 0$ is given by

$$f = \cosh(\ell/2)\mathbb{1} + \sinh(\ell/2)h_\gamma.$$

Proof. $a)$ Let g be an orientation preserving isometry which maps the endpoints at infinity of γ to -1 and 1. Then we have that gfg^{-1} is a hyperbolic isometry with fixed points -1 and 1 and since the displacement length is invariant under conjugation we have that $gfg^{-1} = \tilde{\nu}_\ell$. Then, since $m_\gamma = g^{-1}m_{g(\gamma)}g$, $m_{g(\gamma)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$gfg^{-1} = \tilde{\nu}_\ell = \cosh(\ell/2)\mathbb{1} + \sinh(\ell/2)m_{g(\gamma)}$$

$$\Rightarrow f = g^{-1}(\cosh(\ell/2)\mathbb{1} + \sinh(\ell/2)m_{g(\gamma)})g = \cosh(\ell/2)\mathbb{1} + \sinh(\ell/2)m_\gamma.$$

A similar argument using h_γ instead of m_γ shows $b)$ if $\gamma \subset \mathbb{H}$. $\square$

2.5.3 Parabolic Type

If $A \in \mathbf{PSL}_2(\mathbb{R})$ represents a Möbius transformation with unique fixed point ∞. Then $c = 0$, $d = \frac{1}{a}$, and $a + \frac{1}{a} = 2 \Rightarrow a = d = 1$

$$A = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

for some $b \in \mathbb{R}$. Then it's obvious that any parabolic transformation is conjugate to A.

2.6 The Three-Dimensional Hyperbolic Space

In this section we're not going to repeat all the details analogous as in the two-dimensional case. One starts with the set $\mathbb{H}^3 = \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : \; x_0^2 - x_1^2 - x_2^2 - x_3^2 = 1\}$, and then $\mathbb{H}^3_+ := \mathbb{H}^3 \cap \{x_0 > 0\}$ with line element $ds^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$.

To simplify computations complex numbers can't deal now with all the space. The most intuitive solution is to take hypercomplex numbers and the best are those which are a field, in this particular case a skew-field (a field without the commutativity property) called the quaternions. These numbers are known as the Hamilton's quaternions $\mathcal{H} := \mathcal{H}(-1, -1)$. One usually writes $1,\; i,\; j,\; k$ for the standard $\mathbb{R}$-basis of $\mathcal{H}$. Take $q = a + bi + cj + dk \in \mathcal{H},\; a, b, c, d \in \mathbb{R}$, then its conjugate is $\overline{q} = a - bi - cj - dk$ and its modulus is defined as $|q| = \sqrt{q\overline{q}} = \sqrt{a^2 + b^2 + c^2 + d^2}$.

We proceed as in the two-dimensional case. First, we introduce the unit ball model (analogous to the Poincaré disk model)

$$\pi : \mathbb{H}^3 \to \mathbb{B}, \quad \pi(x_0, x_1, x_2, x_3) = \left(\frac{x_1}{1 + x_0}, \frac{x_2}{1 + x_0}, \frac{x_3}{1 + x_0} \right) =: u_0 + u_1 i + u_2 j,$$

$$\pi^{-1}(u_1, u_2, u_3) = \frac{1}{1 - u_1^2 - u_2^2 - u_3^2}(1 + u_1^2 + u_2^2 + u_3^2, 2u_1, 2u_2, 2u_3) = (x_0, x_1, x_2, x_3),$$

$$u_n := \frac{x_n}{1 + x_0}, \quad n = 1, 2, 3.$$

There's a metric, a measure and a Laplace-Beltrami operator on $\mathbb{B} := \{u = u_1 + u_i + u_3 j : \; |u|^2 = u_1^2 + u_2^2 + u_3^2 < 1\}$ induced by π

$$ds^2 = 4\frac{du_1^2 + du_2^2 + du_3^2}{(1 - u_1^2 - u_2^2 - u_3^2)^2},$$

$$dv = \frac{8 du_1 du_2 du_3}{(1 - u_1^2 - u_2^2 - u_3^2)^3},$$

$$\Delta = \frac{(1 - \rho^2)^2}{4} \left(\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2} + \frac{\partial^2}{\partial u_3^2} \right) + \frac{1 - \rho^2}{2} \left(\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2} + \frac{\partial}{\partial u_3} \right)$$

where $\rho = |u|$. Then the hyperbolic distance in $\mathbb{B}$ has an analogous expression as in the Poincaré disk

$$d(0, u) = \log \frac{1+\rho}{1-\rho}.$$

Definition 2.3. For $q = a + bi + cj + dk \in \mathcal{H}$ with $a, b, c, d \in \mathbb{R}$ we define

$$q^* = a + bi + cj - dk.$$

The map $q \mapsto \overline{q}^*$ is a skew field automorphism and * is a skew field anti-automorphism of $\mathcal{H}$. This two maps are involutory and commute with each other for $q \in \mathcal{H}$.

Definition 2.4. We define

$$\mathbf{SB}_2(\mathcal{H}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{H}) : \ d = \overline{a}^*, \ c = \overline{a}^*, \ a\overline{a} - b\overline{b} = 1 \right\} < M_2(\mathcal{H}).$$

For $u \in \mathbb{B}$ and $f = \begin{bmatrix} a & b \\ \overline{b}^* & \overline{a}^* \end{bmatrix} \in \mathbf{SB}_2(\mathcal{H})$ the transformations $f(u) = (au+b)(\overline{b}^* u + \overline{a}^*)^{-1}$ are isometries of $\mathbb{B}$ and define an action of $\mathbf{SB}_2(\mathcal{H})$ on $\mathbb{B}$ and we also have the isomorphism $\mathrm{Isom}^+(\mathbb{B}) \cong \mathbf{SB}_2(\mathcal{H})$.

The upper-half space $\mathbb{H}$ in Euclidean three-space gives a convenient model of 3-dimensional hyperbolic space. We use the following coordinates:

$$\mathbb{H} := \mathbb{C}\times]0, \infty[= \{(z, r) \in \mathbb{C} \times \mathbb{R} : \ r > 0\} = \{(x, y, r) \in \mathbb{R}^3 : \ r > 0\}.$$

We use the notation:

$$P = (z, r) = (x, y, r) = z + rj,$$

where $z = x + yi$ and $j = (0, 0, 1)$. We can map isometrically the unit ball $\mathbb{B}$ to $\mathbb{H}$ by the Möbius transformation $\varphi : \mathbb{B} \to \mathbb{H}$

$$\varphi(u) = (u + 1)j(1 - u)^{-1}, \ \ \varphi^{-1}(P) = (P - j)(P + j)^{-1}$$

and then the induced metric, measure, and Laplace-Beltrami operator are

$$ds^2 = \frac{dx^2 + dy^2 + dr^2}{r^2},$$

$$dv = \frac{dx\ dy\ dr}{r^3},$$

$$\Delta = r^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial r^2} \right) - r \frac{\partial}{\partial r}.$$

The geodesics are (as in the two-dimensional case) generalized circles which intersect the boundary orthogonally. The hyperbolic planes (also called geodesic hyperplanes) are Euclidean generalized hemispheres (we admit Euclidean planes) which intersect the boundary orthogonally. We have the isomorphism $\mathrm{Isom}^+(\mathbb{H}) \cong \mathbf{PSL}_2(\mathbb{C})$, the action is defined as follows:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{PSL}_2(\mathbb{C}) \Rightarrow MP := (aP+b)(cP+d)^{-1}.$$

To check that this action is well defined in $\mathbb{H}$ (so the element k doesn't appear) we compute the mapped element in coordinates

$$M(z+rj) = \frac{(az+b)\overline{(cz+d)} + a\bar{c}r^2 + rj}{|cP+d|^2} =: z^* + r^* j.$$

The action of an element $M \in \mathbf{PSL}_2(\mathbb{C})$ in the boundary $\partial\mathbb{H} = \mathbb{P}^1_{\mathbb{C}}$ is given by

$$[z,w] \mapsto M[z,w] := [az+bw, cz+dw], \quad (z,w) \in \mathbb{C} \setminus \{(0,0)\}, \ [z,w] \in \mathbb{P}^1_{\mathbb{C}}.$$

We have also the distance formula, taking $p = z + rj$ and $P' = z' + r'j$

$$\delta(P,P') := \cosh d(P,P') = 1 + \frac{|z - z' + (r-r')j|^2}{2rr'}.$$

If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{PSL}_2(\mathbb{C})$ then

$$2\delta(j, Mj) = 2\cosh d(j, Mj) = |a|^2 + |b|^2 + |c|^2 + |d|^2.$$

Proposition 2.7. *The group $\boldsymbol{PSL}_2(\mathbb{C})$ acts in the following sense doubly transitively on $\mathbb{H}$: For all $P, P', Q, Q' \in \mathbb{H}$ such that $d(P,P') = d(Q,Q')$ there exists an $M \in \boldsymbol{PSL}_2(\mathbb{C})$ such that $MP = Q$ and $MP' = Q'$.*

Proof. Let $P = z + rj$, $z \in \mathbb{C}$, $r > 0$. Then the element

$$T_1 := \begin{bmatrix} \frac{1}{\sqrt{r}} & 0 \\ 0 & \sqrt{r} \end{bmatrix} \begin{bmatrix} 1 & -z \\ 0 & 1 \end{bmatrix} \in \mathbf{PSL}_2(\mathbb{C})$$

maps P onto j. There exists an element $T_2 = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \in \mathbf{SU}_2$ (in the stabilizer of j) such that for $T = T_2 T_1$: $TP = j$, $TP' = tj$ with $t \geq 1$. Since $q := T_1 P' = \frac{z'-z+r'j}{r}$, if we suppose $z' \neq z$ (the case $z' = z$ is obvious) we have

$$0 = \frac{(a(z'-z)+br)(ar - b\overline{(z'-z)}) - abr'^2}{r|\bar{a} - \bar{b}q|^2}$$

$$\Rightarrow a^2(z'-z)r - b^2\overline{(z'-z)}r + ab(r^2 - r'^2 - |z'-z|^2) = 0$$

$$\Rightarrow (z'-z)r + w(r^2 - r'^2 - |z'-z|^2) - w^2\overline{(z'-z)}r = 0, \quad w := \frac{b}{a} \ if \ z \neq z'$$

$$\Rightarrow w = \frac{2(r^2 - r'^2 - |z'-z|^2) \pm \sqrt{(r^2 - r'^2 - |z'-z|^2)^2 + 4|z'-z|^2 r^2}}{2\overline{(z'-z)}r} \in \mathbb{C}$$

without loss of generality we can take one of the two roots

$$\Rightarrow T_2 = \begin{bmatrix} a & wa \\ -\overline{wa} & \overline{a} \end{bmatrix}, \quad |a|^2(1+|w|^2) = 1 \Rightarrow t = \frac{r'}{r|a|^2|1-\overline{w}q|^2} = \frac{r'}{r}\frac{1+|w|^2}{|1-\overline{w}q|^2} \in \mathbb{R}_+.$$

Then

$$d(P,P') = d(TP,TP') = d(j,tj) = \int_1^t \frac{dr}{r} = \log t.$$

So then, $t = e^{d(P,P')}$. Similarly there exists an element $S \in \mathbf{PSL}_2(\mathbb{C})$ such that $SQ = j$, $SQ' = tj$ with the same t, since $d(P,P') = d(Q,Q')$. Then $M = S^{-1}T$. □

Definition 2.5. A map $f : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ is called a point-pair invariant if

$$f(MP, MQ) = f(P, Q)$$

for all $P, Q \in \mathbb{H}$, $M \in \mathbf{PSL}_2(\mathbb{C})$.

Proposition 2.8. *A map $f : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ is a point-pair invariant if and only if there exists a map $k : [1, \infty[\to \mathbb{C}$ such that $f = k \circ \delta$.*

Proof. If $f = k \circ \delta$ then f is obviously a point-pair invariant.
Assume that f is a point-pair invariant and let $P, Q \in \mathbb{H}$. Take $M \in \mathbf{PSL}_2(\mathbb{C})$ such that $MP = j$ and $MQ = e^{d(P,Q)}j$ (this is possible by proposition -). Then

$$f(P,Q) = f(MP,MQ) = f(j, e^{d(P,Q)}j) = g(d(P,Q))),$$

where $g(x) = f(j, e^j)$ for $x \geq 0$. Then $k(t) := g(\mathrm{arcosh} t)$ satisfies the required properties. □

Proposition 2.9. *For all $P, Q, R \in \mathbb{H}$ the inequality*

$$\frac{1}{4}\frac{\delta(Q,R)}{\delta(P,Q)} \leq \delta(P,R) \leq 4\delta(P,Q)\delta(Q,R)$$

holds.

Proof. Since δ is a point-pair invariant we only need to check the right-hand side inequality in the special case $Q = j$ (to obtain the left-hand side we inerchange P and Q in the right-hand side of the inequality). We put $P = z + rj,\ R = w + tj$ with $z, w \in \mathbb{C},\ r, t > 0$. Then we have

$$4\delta(P,j)\delta(j,R) = \frac{|z|^2 + r^2 + 1}{r}\ \frac{|w|^2 + t^2 + 1}{t} \geq \frac{|z|^2 + |w|^2 + r^2 + t^2}{rt}$$
$$\geq \frac{|z-w|^2 + r^2 + t^2}{2rt} = \delta(P,R).$$

□

2.6.1 Discontinuous Groups

Definition 2.6. A subgroup $\Gamma < \mathbf{PSL}_2(\mathbb{C})$ is called a discontinuous group if and only if for every $P \in \mathbb{H}$ and for every sequence $(T_n)_{n\geq 1}$ of distinct elements of Γ the sequence $(T_nP)_{n\geq 1}$ has no accumulation point in $\mathbb{H}$. We say that Γ acts discontinuously on $\mathbb{H}$.

A subgroup $\Gamma < \mathbf{PSL}_2(\mathbb{C})$ is called discrete if and only if its inverse image in $\mathbf{SL}_2(\mathbb{C}) \subset \mathbb{C}^4$ is discrete in the vector space topology. The topology on $\mathbf{SL}_2(\mathbb{C}) \subset \mathbb{C}^4$ can be defined by the following norm

$$\|A\| := \sqrt{|a|^2 + |b|^2 + |c|^2 + |d|^2} = \sqrt{2\delta(j, Aj)}, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The norm defines also a topology on $\mathbf{PSL}_2(\mathbb{C})$.

Theorem 2.5. *A subgroup $\Gamma < \boldsymbol{PSL}_2(\mathbb{C})$ is a discontinuous group if and only if is discrete in $\boldsymbol{PSL}_2(\mathbb{C})$*

Proof. If Γ contains an infinite convergent sequence $(T_n)_{n\in\mathbb{N}}$ then there exists a sequence of points $P_n := T_nj$ which converge in $\mathbb{H}$ and then Γ is not discontinuous. Conversely, if Γ is not discontinuous in $\mathbb{H}$ then there exist $P \in \mathbb{H}$ such that the orbit $\Gamma(P) := \{T(P) \in \mathbb{H} : T \in \Gamma\}$ has an accumulation point. Then there exist some $P_n := T_nP$, $n \in \mathbb{N}$ where $T_n \in \Gamma$ are of the form $\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$. Then the numbers $\ell_n := d(j, T_nj)$ define a bounded sequence

$$2\cosh \ell_n = a_n^2 + b_n^2 + c_n^2 + d_n^2$$

which shows that the sequence T_n is contained in a ball of finite radius in $\mathbb{C}^4$ and then Γ is not a discrete group. $\square$

As in the two dimensional case we divide the transformations of $\mathbf{PSL}_2(\mathbb{C})$ into diferent classes.

Definition 2.7. An element $\gamma \in \mathbf{SL}_2(\mathbb{C})$, $\gamma \neq \mathbb{1}$ is called

$$\text{parabolic if } |\mathrm{tr}(\gamma)| = 2 \text{ and } \mathrm{tr}(\gamma) \in \mathbb{R},$$

$$\text{hyperbolic if } |\mathrm{tr}(\gamma)| > 2 \text{ and } \mathrm{tr}(\gamma) \in \mathbb{R},$$

$$\text{elliptic if } 0 \leq |\mathrm{tr}(\gamma)| < 2 \text{ and } \mathrm{tr}(\gamma) \in \mathbb{R},$$

$$\text{loxodromic if } \mathrm{tr}(\gamma) \in \mathbb{C} \setminus \mathbb{R}.$$

Then an element of $\mathbf{PSL}_2(\mathbb{C})$ is called parabolic, hyperbolic, elliptic or loxodromic if its preimage in $\mathbf{SL}_2(\mathbb{C})$ has the corresponding preperty.

Proposition 2.10. *Let $\gamma \in \boldsymbol{SL}_2(\mathbb{C})$, $\gamma \neq \mathbb{1}$*

$$\textit{parabolic iff it is conjugate to } \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \textit{ in } \boldsymbol{PSL}_2(\mathbb{C}),$$

$$\textit{hyperbolic iff it is conjugate to } \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \textit{ in } \boldsymbol{PSL}_2(\mathbb{C}) \textit{ with } \lambda \in \mathbb{R}_+,$$

$$\textit{elliptic iff it is conjugate to } \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \textit{ in } \boldsymbol{PSL}_2(\mathbb{C}) \textit{ with } \lambda \in \mathbb{S}^1 \subset \mathbb{C},$$

$$\textit{loxodromic iff it is conjugate to } \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \textit{ in } \boldsymbol{PSL}_2(\mathbb{C}) \textit{ with } \lambda \in \mathbb{C} \setminus (\mathbb{R} \cup \mathbb{S}^1).$$

Proof. First calculate the eigenvalues

$$\det(\gamma - \lambda\mathbb{1}) = \lambda^2 - \operatorname{tr}(\gamma)\lambda + 1 \implies \lambda_\pm = \frac{\operatorname{tr}(\gamma) \pm \sqrt{(\operatorname{tr}(\gamma))^2 - 4}}{2}.$$

Then γ is hyperbolic iff $\lambda := \lambda_+ \in \mathbb{R}$ and one has $\lambda_- = \lambda^{-1}$ (since we don't distinguish the sign of the matrix we can suppose $\lambda > 0$). In the elliptic case one has $|\lambda| = |\lambda^{-1}|,\ \lambda \in \mathbb{C}$ (since $\lambda_- = \overline{\lambda}$) which implies $|\lambda| = 1$ (the equivalence is obvious).
In the parabolic case one has $\lambda_+ = \lambda_- = 1$, we can suppose $b \neq 0$ which implies $c = -\frac{(a-1)^2}{b}$ and

$$\gamma - \mathbb{1} = \begin{bmatrix} a-1 & b \\ -\frac{(a-1)^2}{b} & 1-a \end{bmatrix} \implies \ker(\gamma - \mathbb{1}) = \langle(-b, a-1)\rangle,$$

choosing $V = \frac{1}{\sqrt{b}} \begin{bmatrix} -b & 0 \\ a-1 & -1 \end{bmatrix} \in \mathbf{PSL}_2(\mathbb{C})$ we have

$$V^{-1}\gamma V = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

In the loxodromic case one has a diagonal matrix diferent from all the other cases. □

Corollary 2.5.1. *Let $\gamma \neq \mathbb{1}$ be an element of $\boldsymbol{PSL}_2(\mathbb{C})$. Then*

1. *γ is parabolic iff has a unique fixed point in $\mathbb{P}^1_{\mathbb{C}}$.*

2. *γ is elliptic iff it has two fixed points in $\mathbb{P}^1_{\mathbb{C}}$ and if the points on the geodesic line joining the two fixed points are left fixed. γ is then a rotation around this line.*

3. *γ is hyperbolic iff it has two fixed points in $\mathbb{P}^1_{\mathbb{C}}$ and if any circle in $\mathbb{P}^1_{\mathbb{C}}$ through these two points together with its interior is left invariant. The geodesic line in $\mathbb{H}$ joining the two fixed points is left invariant but γ has no fixed points in $\mathbb{H}$.*

4. *γ is loxodromic in all other cases. γ has then two fixed points in $\mathbb{P}^1_{\mathbb{C}}$ and no fixed point in $\mathbb{H}$. The geodesic joining the two fixed points is left invariant. γ may leave the circles joining the two fixed points invariant interchanging exterior and interior.*

Let's try to understand in a better way how this transformations act on $\mathbb{H}$. Take $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{PSL}_2(\mathbb{C})$ $\gamma \neq \mathbb{1}$. If γ is hyperbolic then there exists $V \in \mathbf{PSL}_2(\mathbb{C}),\ \ell > 0$ such that

$$\tilde{\gamma} := V^{-1}\gamma V = \begin{bmatrix} e^{\ell/2} & 0 \\ 0 & e^{-\ell/2} \end{bmatrix} \implies |\mathrm{tr}(\gamma)| = |\mathrm{tr}(\tilde{\gamma})| = 2\cosh(\ell/2),$$

$$2\delta(j, \tilde{\gamma}j) = 2\cosh(d(j, \tilde{\gamma}j)) = 2\cosh(\ell) \implies \ell = d(j, \tilde{\gamma}j) = d(P, \gamma P)$$

where $P = Vj$ is a point which lies on the axis of γ. Let $Q \in \mathbb{H}$ be any point and $Q' = V^{-1}Q = z' + r'j,\ z' \in \mathbb{C},\ r' > 0$

$$d(Q, \gamma Q) = d(Q', \tilde{\gamma}Q') = \frac{|1 - e^{\ell}|^2 |z'|^2 + (1 + e^{2\ell})r'^2}{2e^{\ell} r'^2} > \frac{1 + e^{2\ell}}{2e^{\ell}} = \cosh(\ell).$$

Then Theorem 2.4 is also true in dimension 3. $\ell = \ell_\gamma$ is called the *displacement length* of γ.
If γ is elliptic we have essentially the same (except the distance properties). Just calling $\ell = i\varphi,\ \varphi \in]-\pi, \pi]$ we obtain $|\mathrm{tr}(\gamma)| = \cos(\varphi/2)$ where φ is called the *angle of rotation.*
If γ is loxodromic then there exist $V \in \mathbf{PSL}_2(\mathbb{C})$ such that

$$\tilde{\gamma} := V^{-1}\gamma V = \begin{bmatrix} e^{\ell/2 + i\varphi/2} & 0 \\ 0 & e^{-\ell/2 - i\varphi/2} \end{bmatrix} = \begin{bmatrix} e^{\ell/2} & 0 \\ 0 & e^{-\ell/2} \end{bmatrix} \begin{bmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{bmatrix},$$

$$|\mathrm{tr}(\gamma)| = |\mathrm{tr}(\tilde{\gamma})| = 2|\cosh(\ell/2 + i\varphi/2)| = 2|\cosh(\ell)\cos(\varphi/2) + i\sinh(\ell/2)\sin(\varphi/2)|.$$

Joining the properties of the elliptic and the hyperbolic elements (since a loxodromic element can be decomposed in a product of an elliptic and a hyperbolic element) we deduce that a loxodromic element has a displacement length $\ell > 0$ and a rotation angle $\varphi \in]-\pi, \pi]$.

Definition 2.8. A closed subset $\mathcal{F} \subset \mathbb{H}$ is called a fundamental domain of the discontinuous group $\Gamma < \mathbf{PSL}_2(\mathbb{C})$ if

1. $\mathcal{F}$ meets each Γ-orbit at least once,
2. the interior $\mathring{\mathcal{F}}$ meets each Γ-orbit at most once,
3. $\partial\mathcal{F}$ has zero Lebesgue measure.

There is a construction of the fundamental domain of a discontinuous group $\Gamma < \mathbf{PSL}_2(\mathbb{C})$ given by the Poincaré normal polyhedron or Dirichlet fundamental domain.

Definition 2.9. Let $\Gamma < \mathbf{PSL}_2(\mathbb{C})$ be a discontinuous group. Let $Q \in \mathbb{H}$ such that $\gamma Q \neq Q$ for all $\gamma \in \Gamma \setminus \{\mathbb{1}\}$. Then the Poincaré normal polyhedron for Γ of center Q is

$$\mathcal{P}_Q(\Gamma) := \{P \in \mathbb{H} : \ d(P, Q) \leq d(\gamma P, Q) \ \forall \gamma \in \Gamma\}.$$

Since Γ is a countable set, there exists a $Q \in \mathbb{H}$ such that $\gamma Q \neq Q$ for all $\gamma \in \Gamma \setminus \{\mathbb{1}\}$.
If $M \subset \mathbb{H}$ is a Borel-measurable set we also use the notation

$$v(M) = \int_M dv.$$

Definition 2.10. Let $\Gamma < \mathbf{PSL}_2(\mathbb{C})$ be a discontinuous group and $\mathcal{F} \subset \mathbb{H}$ a fundamental domain for Γ. Then Γ is of finite covolume or a cofinite group if

$$\mathrm{vol}(\Gamma) = \int_{\mathcal{F}} dv < \infty.$$

$\mathrm{vol}(\Gamma)$ is called the covolume of Γ. Γ is cocompact if Γ has a compact fundamental domain. $\Gamma < \mathbf{SL}_2(\mathbb{C})$ is called cofinite or cocompact if its image in $\mathbf{PSL}_2(\mathbb{C})$ has this property.

The covolume of a discontinuous group is well defined in the following sense

Proposition 2.11. *Let $\Gamma < \boldsymbol{PSL}_2(\mathbb{C})$ be a discontinuous group and let $\mathcal{F}_1, \mathcal{F}_2$ be two fundamental domains for Γ. If $\int_{\mathcal{F}_1} dv < \infty$ then $\int_{\mathcal{F}_2} dv < \infty$ and $\int_{\mathcal{F}_1} dv = \int_{\mathcal{F}_2} dv$.*

Proof. Since $(\mathcal{F}_1 \cap \gamma_1 \mathcal{F}_2) \cap (\mathcal{F}_1 \cap \gamma_2 \mathcal{F}_2)$ has zero Lebesgue measure for any pair $\gamma_1, \gamma_2 \in \Gamma$ with $\gamma_1 \neq \gamma_2$

$$v(\mathcal{F}_1) = v(\mathcal{F}_1 \cap (\cup_{\gamma \in \Gamma} \gamma \mathcal{F}_2)) = \sum_{\gamma \in \Gamma} v(\mathcal{F}_1 \cap \gamma \mathcal{F}_2) = \sum_{\gamma \in \Gamma} v(\gamma^{-1} \mathcal{F}_1 \cap \mathcal{F}_2) = v(\mathcal{F}_2).$$

□

Let $B_r(Q) := \{P \in \mathbb{B} : \ d(P, Q) < r\}$ be the ball of radius $r > 0$ and center $Q \in \mathbb{H}$. We set $r = \frac{1+R}{1-R}$ and we get

$$\begin{aligned} v(B_r(0)) &= 8 \int_{0 \leq \rho \leq R} \frac{du_0 \ du_1 \ du_2}{(1 - u_0^2 - u_1^2 - u_2^2)^3} = 32\pi \int_0^R \frac{\rho^2}{(1-\rho^2)^3} \ d\rho \\ &= 4\pi \left(\frac{R(1+R^2)}{(1-R^2)^2} - \frac{1}{2} \log \frac{1+R}{1-R} \right) \\ &= 2\pi(\sinh(r)\cosh(r) - r) = \pi(\sinh(2r) - 2r). \end{aligned}$$

Which implies

$$B_r(Q) \sim \frac{4}{3}\pi r^3, \quad \text{as } r \to 0$$

and on the other hand

$$v(B_r(Q)) \sim \frac{\pi}{2}e^{2r}, \quad \text{as } r \to \infty.$$

We also have the inequality

$$v(B_r(Q)) \leq \pi \sinh(2r) \leq \frac{\pi}{2}e^{2r} \quad \forall r \geq 0.$$

Theorem 2.6. *Let $\Gamma < \boldsymbol{PSL}_2(\mathbb{C})$ be a discrete group of finite covolume. Then, Γ contains a parabolic element if and only if Γ is not cocompact.*

If $\Gamma < \mathbf{PSL}_2(\mathbb{C})$ is a discontinuous group, acting fixed-point freely on $\mathbb{H}$ then the quotient space $\Gamma \setminus \mathbb{H}$ inherits from $\mathbb{H}$ the structure of an orientable Riemannian manifold. The projection $\pi_\Gamma : \mathbb{H} \to \Gamma \setminus \mathbb{H}$ is a local isometry. Γ is cocompact iff $\Gamma \setminus \mathbb{H}$ is compact, and is cofinite iff $\Gamma \setminus \mathbb{H}$ is of finite volume. In this case we have $\mathrm{vol}(\Gamma) = \int_{\Gamma\setminus\mathbb{H}} dv$, where dv is the volume form coming from the Riemannian structure.

Theorem 2.7. *Let $\Gamma < \boldsymbol{PSL}_2(\mathbb{C})$ be a discontinuous cocompact group and let $\pi_1(\Gamma \setminus \mathbb{H})$ be the fundamental group of the Riemannian manifold $\Gamma \setminus \mathbb{H}$. Then we have*

$$\Gamma \cong \pi_1(\Gamma \setminus \mathbb{H}).$$

The proof of this result is the same as the case of $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$ (see [25]).
Let $T \in \Gamma$ then, we denote by $\{T\}_\Gamma$ the Γ-conjugacy class. We denote by ℓ_T the displacement length of T which is greater than 0 iff T is a hyperbollic or loxodromic element.

Lemma 2.8. *Suppose that $\Gamma < \boldsymbol{PSL}_2(\mathbb{C})$ is a discrete cocompact group. Then the counting function*

$$\pi_0(x) := |\{\{T\}_\Gamma : \ T \in \Gamma, \ 0 < \ell_T \leq \log x\}|$$

satisfies the growth restriction $\pi_0(x) = O(x^2)$ as $x \to \infty$.

Proof. Suppose $\ell_T > 0$ then we have $\ell_T = \inf\{d(P, TP) : \ P \in \mathbb{H}\}$ and $\ell_T = d(P, TP)$ iff P lies on the axis of T. Take P a point on the axis of T. Then there exists $S \in \Gamma$ such that $Q := SP \in \mathcal{F}$ and

$$\ell_T = d(P, TP) = d(SP, STS^{-1}(SP)) = d(Q, STS^{-1}Q),$$

with $V := STS^{-1} \in \{T\}_\Gamma$. Let $d(A, B) := \inf\{d(P, Q) : \ P \in A, \ Q \in B\}$ be the distance between two non-empty sets $A, B \subset \mathbb{H}$, and $d_0 := \sup\{d(P, Q) : \ P, Q \in \mathcal{F}\}$ be the hyperbolic diameter of $\mathcal{F}$. Choose a point $P_0 \in \mathcal{F}$. Then

$$\begin{aligned}\pi_0 = |\{\{T\}_\Gamma : \ 0 < \ell_T \leq \log x\}| &\leq |\{V \in \Gamma : \ d(\mathcal{F}, V\mathcal{F}) \leq \log x\}| \\ \leq |\{V \in \Gamma : \ d(P_0, VP_0) \leq \log x + 2d_0\}| &\leq |\{V \in \Gamma : \ v\mathcal{F} \subset B_{\log x + 3d_0}(P_0)\}| \\ \leq \frac{v(B_{\log x+3d_0}(P_0))}{v(\mathcal{F})} &< \frac{\pi}{2v(\mathcal{F})}e^{6d_0}x^2.\end{aligned}$$

□

3 Automorphic functions

In this chapter we will introduce some functions, operators and transforms which will be useful in the next chapter to compute the trace of the Laplace-Beltrami operator.

In the first section we will introduce some functions $f : \mathbb{H} \to \mathbb{C}$ which still invariant on the group action. A natural way to define this functions is to consider the Poincaré summation process

$$f(P) = \sum_{M \in \Gamma} h(MP)$$

for some suitable function $h : \mathbb{H} \to \mathbb{C}$. Of particular interest is to take point-pair invariants, for example functions of the hyperbolic distance like $\delta(P,Q)^{-1-s}$, for $s \in \mathbb{C}$ or $\mathrm{e}^{-t\delta(P,Q)}$, for $t \in \mathbb{R}_+$.

In the second section we will prove that the resolvent operator of the unique extension $\widetilde{\Delta}$ of the Laplace-Beltrami operator Δ, $R_\lambda = (-\widetilde{\Delta} - \lambda)^{-1}$ is described by an integral operator whose kernel is the Mass-Selberg series $F(P,Q,s)$. We leave to the reader the basic aspects of how does the Laplacian operates on its natural domain, the Hilbert space $L^2(\Gamma \setminus \mathbb{H})$ for discrete cocompact groups $\Gamma < \mathbf{PSL}_2(\mathbb{C})$ (for cocompact and for finite covolume cases see [7]).

In this chapter we follow [7], all proofs given are from the reference and depending on the situation we extend the proof to make it easy to read and understand or we skip some details to go further.

3.1 Poincaré series and Eisenstein series

Definition 3.1. For every $P, Q \in \mathbb{H}$ we define the series

$$H(P,Q,s) = \sum_{M \in \Gamma} \delta(P,MQ)^{-1-s}$$

which is Γ-invariant in both variables P and Q.

The series H has an absolute abscissa of convergence σ_0 (we call this the abscissa of convergence of Γ too) such that the series converges uniformly in the compact subsets of $\{(P,Q,s) \in \mathbb{H} \times \mathbb{H} \times \mathbb{C} \mid \Re s > \sigma_0\}$. Note that by a previous result we have that σ_0 does not depend on P, Q. Then, the series defines a holomorphic function of the variable s in the half-plane $\Re s > \sigma_0$.

Proposition 3.1. *The abscissa of convergence of the defined series H satisfies $\sigma_0 \leqslant 1$.*

Proof. Let $K \subset \mathbb{H} \times \mathbb{H} \times \{\ s \in \mathbb{C} \mid \Re s > 1\}$ be a compact subset and

$$\sigma := \sigma_K := \min\{\ \Re s \mid (P,Q,s) \in K \quad \forall P, Q \in K : H(P,Q,s) < +\infty\}.$$

Then for every $(P, Q, s) \in K$ and $M \in \Gamma$ we have:

$$\begin{aligned}\delta(P, MQ)^{-1-\Re s} &\leqslant 4^{1+\sigma}\delta(P, P_0)^{1+\sigma}\delta(P_0, MQ)^{-1-\sigma}\\ &= 4^{1+\sigma}\delta(P, P_0)^{1+\sigma}\delta\left(M^{-1}P_0, Q\right)^{-1-\sigma}\\ &\leqslant 4^{2+2\sigma}\delta(P, P_0)^{1+\sigma}\delta(Q, Q_0)^{1+\sigma}\delta(P_0, MQ_0)^{-1-\sigma}.\end{aligned}$$

So we just need to prove it for some $P_0, Q_0 \in \mathbb{H}$.
If $\mathcal{F}$ is a measurable fundamental domain for Γ then by the monotone convergence theorem:

$$\int_{\mathcal{F}} H(P, j, \sigma)\, dv(P) = \int_{\mathcal{F}} \sum_{M\in\Gamma} \delta(MP, j)^{-1-\sigma}\, dv(P) = \sum_{M\in\Gamma} \int_{M\mathcal{F}} \delta(P, j)^{-1-\sigma}\, dv(P)$$

$$= \int_{\mathbb{H}} \delta(P, j)^{-1-\sigma}\, dv(P) = \int_0^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left(\frac{2r}{x^2+y^2+r^2+1}\right)^{1+\sigma} \frac{dx\, dy\, dr}{r^3}$$

$$= 2^{2+\sigma}\pi \int_0^{+\infty}\int_0^{+\infty} \frac{\rho}{(\rho^2+r^2+1)^{1+\sigma}}\frac{d\rho\, dr}{r^{2-\sigma}} = \frac{2^{1+\sigma}\pi}{\sigma}\int_0^{+\infty} \frac{1}{(r^2+1)^{\sigma}}\frac{dr}{r^{2-\sigma}}.$$

But for $\sigma = 0$ we have that the integral $\int_0^{+\infty} \frac{1}{r^2}\, dr$ is divergent and for $\sigma = 2$ $\int_0^{+\infty} \frac{1}{(1+r^2)^2}\, dr < \frac{\pi}{2}$ so we have $0 < \sigma \leqslant 2$. Then:

$$\begin{aligned}0 < \int_0^{+\infty} \frac{1}{(r^2+1)^{\sigma}}\frac{dr}{r^{2-\sigma}} &= \int_0^{1} \frac{1}{(r^2+1)^{\sigma}}\frac{dr}{r^{2-\sigma}} + \int_1^{+\infty} \frac{1}{(r^2+1)^{\sigma}}\frac{dr}{r^{2-\sigma}}\\ &< \int_0^{1} \frac{1}{(r^2+1)^{\sigma}}\frac{dr}{r^{2-\sigma}} + \frac{\pi}{2} < \int_0^1 r^{\sigma-2}\, dr + \frac{\pi}{2}.\end{aligned}$$

The last integral is finite for $\sigma > 1$ so we have that $H(P, j, \sigma)$ converges for almost all P and then exist some $P_0 \in \mathbb{H}$ such that $H(P_0, j, \sigma)$ converges. $\square$

If we use the information on the behaviour of the orbits of Γ by the map

$$\pi(P, Q, x) = |\{\, M \in \Gamma \mid \delta(P, MQ) < x\}|$$

(this function is defined in [7] where some properties are proved, since we are dealing with discrete cocompact groups we can restrict to π_0 defined in Lemma 2.8) we obtain the following results:

Proposition 3.2. *Let Γ be a discrete subgroup of $\boldsymbol{PSL}_2(\mathbb{C})$ and $K \subset \mathbb{H}$ be a compact set. Then there exist a constant $C_1(K)$ depending only on Γ and K such that*

$$H(P, Q, s) \leqslant C_1(K)\,\frac{s+1}{s-1} \quad \forall P \in K,\ Q \in \mathbb{H},\ s > 1.$$

If Γ has finite covolume then there exists a constant $C_2(K)$ depending only on K and Γ such that

$$H(P, Q, s) \geqslant C_2(K)\,\frac{s+1}{s-1} \quad \forall P \in K,\ Q \in \mathbb{H},\ s > 1.$$

If we transform $\mathbb{H}$ to $\mathbb{B}$ where $P \mapsto 0,\ \ Q \mapsto x \in \mathbb{B}$ we have by the formula of the distance in $\mathbb{B}$

$$\begin{aligned}\delta(P, MQ) &= \cosh d(P, MQ) = \cosh d'(0, g(x)) \\ &= \cosh\left(\log \frac{1 + \|g(x)\|}{1 - \|g(x)\|}\right) = \frac{1 + \|g(x)\|^2}{1 - \|g(x)\|^2}.\end{aligned}$$

Where $g \in \mathit{\Gamma}'$ is the image of M in $\mathit{\Gamma}'$, and $\mathit{\Gamma}'$ is the image in $\mathbb{B}$ of $\mathit{\Gamma}$. Then

$$H(P, Q, s) = \sum_{g \in \mathit{\Gamma}'} \frac{1 + \|g(x)\|^2}{1 - \|g(x)\|^2} =: F(x, s).$$

Theorem 3.1. *Fix $P \in \mathbb{H}$ and $s \in \mathbb{R},\ s > 1$. Then the function $Q \mapsto H(P, Q, s)$ attains its maximum in the closed hyperbollic ball*

$$\left\{\ Q \in \mathbb{H}\ :\ \delta(P, Q) \leqslant \sqrt{\frac{s+2}{s-1}}\ \right\}.$$

Proposition 3.3. *The series H satisfies the following partial differential equation for all $s \in \mathbb{C}$ such that $\Re s > \sigma_0$ where σ_0 is the abscissa of convergence of $\mathit{\Gamma}$.*

$$(-\mathit{\Delta} - (1 - s^2))H(P, Q, s) = (s + 1)(s + 2)H(P, Q, s + 2).$$

Proof. We first transform $\mathbb{H}$ onto $\mathbb{B}$ such that $P \mapsto 0$ and $Q \mapsto x$. Since $\mathit{\Delta}$ commutes with the action of $\mathbf{Iso}^+(\mathbb{B})$

$$\mathit{\Delta}\left(\frac{1 + \|g(x)\|^2}{1 - \|g(x)\|^2}\right)^{1+s} = \left(\mathit{\Delta}\left(\frac{1 + \|x\|^2}{1 - \|x\|^2}\right)^{1+s}\right)\Bigg|_{x \mapsto g(x)}.$$

If we call $\rho := \|x\|$ then the radial part of $\mathit{\Delta}$ is given by:

$$\frac{1}{4}(1 - \rho^2)^2\left(\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\rho^2\frac{\partial}{\partial\rho} + \frac{2\rho}{1 - \rho^2}\frac{\partial}{\partial\rho}\right).$$

Then:

$$\mathit{\Delta}\left(\frac{1 - \rho^2}{1 + \rho^2}\right)^{1+s} = (s + 1)\left(\frac{1 - \rho^2}{1 + \rho^2}\right)^{3+s}\left((s - 1)\left(\frac{1 - \rho^2}{1 + \rho^2}\right)^{-2} - (s + 2)\right).$$

$$\Longrightarrow (\mathit{\Delta} + (1 - s^2))\left(\frac{1 - \rho^2}{1 + \rho^2}\right)^{1+s} = -(s + 1)(s + 2)\left(\frac{1 - \rho^2}{1 + \rho^2}\right)^{3+s}.$$

□

Now we will introduce an analogue of the Jacobi's theta function.

Definition 3.2. We define the series

$$\Theta(P,Q,t) := \sum_{M\in\Gamma} e^{-t\delta(P,MQ)} = \sum_{M\in\Gamma} e^{-t\delta_M}$$

for $P,Q \in \mathbb{H},\ t>0$. We put $\delta_M := \delta(P,MQ)$.

Proposition 3.4. *The series $\Theta(P,Q,t)$ is Γ-invariant with respect to P and with respect Q and it is normally convergent in the whole of* $\mathbb{H}\times\mathbb{H}\times]0,\infty[$.

Proposition 3.5. *For $\Re s > 1$ we have*

$$H(P,Q,s) = \frac{1}{\Gamma(s+1)} \int_0^\infty t^{s-1}\Theta(P,Q,t)\,dt$$

where the integral is absolutely convergent.

Proposition 3.6. *If $K\subset\mathbb{H}$ is a compact subset, there exist constants $c_1,c_2,\varepsilon_1,\varepsilon_2>0$ such that:*

$$c_1 e^{-\varepsilon_1 t} \leqslant \Theta(P,Q,t) \leqslant c_2 e^{-\varepsilon_2 t}$$

for all $t\geqslant 1$ uniformly with respect to $(P,Q)\in K\times K$. There also exists a constant $C_1>0$ such that for all $P\in K$, $Q\in\mathbb{H}$ and $0<t\leqslant 1$

$$\Theta(P,Q,t) \leqslant \frac{C_1}{t^2}.$$

If Γ has finite covolume and $K\subset\mathbb{H}\times\mathbb{H}$ is a compact set, there exists a constant $C_2>0$ such that

$$\frac{C_2}{t^2} \leqslant \Theta(P,Q,t)$$

for all $(P,Q)\in K$ and $0<t\leqslant 1$.

Proposition 3.7. *If $\mathcal{F}$ is a fundamental domain for Γ, we have for all $t>0$*

$$\int_{\mathcal{F}} \Theta(P,Q,t)\,dv(Q) = \frac{4\pi}{t}K_1(t),$$

where K_1 is the usual modified Bessel function.

Proof. $\forall t>0$,

$$\int_{\mathcal{F}} \Theta(P,Q,t)\,dv(Q) = \int_{\mathbb{H}} e^{-t\delta(P,Q)}\,dv(Q) = \int_{\mathbb{H}} e^{-t\delta(j,Q)}\,dv(Q),$$

since δ is a point-pair invariant and v is a $\mathbf{PSL_2}(\mathbb{C})$-invariant. Hence we obtain

$$\int_{\mathcal{F}} \Theta(P,Q,t)\,dv(Q) = \int_0^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-t\left(\frac{x^2+y^2+r^2+1}{2r}\right)}\,\frac{dx\ dy\ dr}{r^3}$$

$$= 2\pi\int_0^{+\infty}\int_0^{+\infty} e^{-t\left(\frac{\rho^2+r^2+1}{2r}\right)}\rho\,\frac{d\rho\ dr}{r^3} = \frac{2\pi}{t}\int_0^{+\infty} e^{\frac{-t}{2}\left(r+\frac{1}{r}\right)}\,\frac{dr}{r^2} = \frac{4\pi}{t}K_1(t).$$

□

Proposition 3.8. *For $P, Q \in \mathbb{H}$, $t > 0$, $\Theta(P,Q,t)$ is a solution of the differential equation*

$$\Delta\ \Theta(P,Q,t) = \left(t^2\frac{\partial^2}{\partial t^2} + 3t\frac{\partial}{\partial t} - t^2\right)\Theta(P,Q,t)$$

where Δ operates on P. The same equation holds when Δ operates on Q.
If φ is a function of δ such that satisfies

$$-\Delta\ \varphi(\delta(P,Q)) = \lambda\ \varphi(\delta(P,Q))$$

then it satisfies a special case of Riemann's differential equation, namely

$$(\delta^2-1)\varphi''(\delta) + 3\delta\varphi'(\delta) + \lambda\ \varphi(\delta) = 0$$

with $\lambda := 1 - s^2$.

For $s \neq 0$ the above equation has the fundamental system of solutions

$$\varphi_s(\delta) = \frac{1}{4\pi}\frac{(\delta+\sqrt{\delta^2-1})^{-s}}{\sqrt{\delta^2-1}} \qquad \textit{and} \qquad \varphi_{-s}$$

which are defined for $\delta > 1$ ([22], [7]).
Note that, since $e^d = \delta + \sqrt{\delta^2-1}$

$$\varphi_s(\delta(P,MQ)) = \frac{1}{4\pi}\frac{e^{-sd(P,MQ)}}{\sinh(d(P,MQ))}, \quad \forall P,Q \in \mathbb{H},\ M \in \Gamma.$$

Definition 3.3. For $P, Q \in \mathbb{H}, P \neq Q \mod \Gamma$ the Maass-Selberg series is defined by

$$F(P,Q,s) := \sum_{M\in\Gamma}\varphi_s(\delta(P,MQ))$$

provided that this series converges absolutely.

Proposition 3.9. *The Maass-Selberg series converges uniformly on compact subsets of* $(\mathbb{H}\times\mathbb{H}\setminus\{\ (P,Q)\in\mathbb{H}|\ P\in\mathbb{H},\ M\in\Gamma\ \})\times\{\ s\in\mathbb{C}\mid\Re s>1\ \}$.

Proposition 3.10. *Let σ_0 be the abscissa of convergence of Γ. For $\Re s > \sigma_0$, P, $Q \in \mathbb{H}$, $P \neq Q \mod \Gamma$, the Maass-Selberg series satisfies the differential equation*

$$-\Delta\ F(P,Q,s) = (1-s^2)\ F(P,Q,s).$$

Definition 3.4. For $P, Q \in \mathbb{H}$, $P \neq Q \mod \Gamma$, $t > 0$ define

$$\Lambda(P,Q,t) := \sum_{M\in\Gamma}\frac{1}{\sqrt{\delta_M^2-1}}e^{-t\left(\delta_M+\sqrt{\delta_M^2-1}\right)}.$$

Proposition 3.11. *For $\Re s > 1$ we have*

$$F(P,Q,s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \Lambda(P,Q,t)\, dt.$$

And the following properties hold uniformly on compact subsets of the corresponding ranges definition.
The series for Λ converges on

$$(\mathbb{H} \times \mathbb{H} \setminus \{\ (P,Q) \in \mathbb{H} |\ P \in \mathbb{H},\ M \in \Gamma\ \}) \times]0,\infty[.$$

There exists constants $c_1, c_2, \varepsilon_1, \varepsilon_2 > 0$ such that for all $t \geqslant 1$

$$c_1 e^{-t\varepsilon_1} \leqslant \Lambda(P,Q,t) \leqslant c_2 e^{-t\varepsilon_2}.$$

There exists a constant $C_1 > 0$ such that for all $t \in]0,1[$

$$\Lambda(P,Q,t) \leqslant \frac{C_1}{t}.$$

If Γ has finite covolume, there exists a constant $C_2 > 0$ such that for all $t \in]0,1[$

$$\Lambda(P,Q,t) \geqslant \frac{C_2}{t}.$$

Now we give a first approach to the automorphic functions which are constructed by averaging the functions on $\mathbb{H}$ defined by

$$z + rj \mapsto r^s.$$

For all $M \in \mathbf{PSL}_2(\mathbb{C})$, $P \in \mathbb{H}$ we write

$$MP = z(MP) + r(MP)j$$

Definition 3.5. Let $\zeta \in \mathbb{P}^1\mathbb{C}$ be a cusp of Γ. For $P \in \mathbb{H}$ we define the formal power series

$$E_A := \sum_{M \in \Gamma'_\zeta \backslash \Gamma} r(AMP)^{1+s}.$$

Which is Γ-invariant whenever it converges absolutely. We call E_A the Eisenstein series for Γ at the cusp ζ.

Take $A \in \mathbf{PSL}_2(\mathbb{C})$ such that $A\zeta = \infty$, then remember that ζ is a cusp if there is a full lattice $\Lambda \subset \mathbb{C}$ such that

$$A\Gamma'_\zeta A^{-1} = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \ :\ \lambda \in \Lambda \right\}.$$

Proposition 3.12. *Let $\Gamma < \boldsymbol{PSL_2}(\mathbb{C})$ be a discrete group and let $\zeta = A^{-1}\infty$ with $A \in \boldsymbol{PSL_2}(\mathbb{C})$ be a cusp of Γ.*
(1) If $\Gamma = \Gamma_\zeta$, the Eisenstein series is a finie sum and equal to a constant multiple of $r(AP)^{1+s}$. The abscissa of convergence of the Eisenstein series equals ∞ in this case.
(2) For $\Gamma \neq \Gamma_\zeta$ the Eisenstein series $E_A(P,s)$ converges iff the series $H(P,Q,s)$ converges for some $Q \in \mathbb{H}$. A necessary condition for convergence is that $\Re s > 0$.

Proposition 3.13. *Let $\Gamma < \boldsymbol{PSL_2}(\mathbb{C})$ be a discrete group and let $\zeta = A^{-1}\infty$ be a cusp of Γ. Then the Eisenstein series $E_A(P,s)$ is a real analytic function of p and a holomorphic function of s in $\Re s > \sigma_0$. It satisfies the differential equation*

$$(-\varDelta - (1-s^2))E_A(P,s) = 0.$$

3.1.1 Expansion of Eigenfunctions and the Selberg Transform

Take the Euclidean Laplace operator $\varDelta_e$ on $\mathbb{B}$

$$\varDelta_e = \sum_{j=1}^{3} \frac{\partial^2}{\partial {\xi_j}^2} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho^2\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\varPhi,$$

where $\rho = \sqrt{\xi_1+\xi_2+\xi_3}$ and $\varPhi$ is the Laplace-Beltrami operator on $\mathbb{S}^2$,

$$\varPhi = \frac{1}{2}\sum_{j,k=1}^{3}\left(\xi_j\frac{\partial}{\partial\xi_k} - \xi_k\frac{\partial}{\partial\xi_j}\right)^2.$$

It is known that $\varPhi$ has a complete otrhonormal system of eigenfunctions $Y_{\ell j}$ (the spherical harmonics) where $\ell \in \mathbb{N}$ and $j = 1, ..., 2\ell+1$ with eigenvalues $\mu_\ell = \ell(\ell+1)$ (so with multiplicity $2\ell+1$)

Theorem 3.2. *Let g: $\mathbb{B} \to \mathbb{C}$ be a solution of the differential equation $-\varDelta g = \lambda g$. Then g has an expansion of the form*

$$g(x) = \sum_{\ell=0}^{\infty}\sum_{j=1}^{2\ell+1} \alpha_{\ell j}\rho^\ell(1-\rho^2)^{1+s}F\left(\ell+s+1, s+1/2; \ell+3/2; \rho^2\right)Y_{\ell j}\left(\frac{x}{\rho}\right)$$

where $x \in \mathbb{B}$, $\rho = \|x\|$. The $\alpha_{\ell j}$ are constants and $F(a,b;c;z)$ is the hypergeometric function.

Observe that all summands with $\ell > 0$ vanish at $x = 0$. Hence the zeroth summand is equal to

$$g_0(\rho) := g(0)(1-\rho^2)^{1+s}F\left(s+1, s+\frac{1}{2}; \frac{3}{2}; \rho^2\right).$$

Then (by [22] page 39)

$$g_0(\rho) = \frac{1}{4}g(0)\frac{1-\rho^2}{\rho}\frac{1}{s}\left(\left(\frac{1+\rho}{1-\rho}\right)^s - \left(\frac{1-\rho}{1+\rho}\right)^s\right),$$

where the right-hand side has to be replaced by its limit at $s \to 0$ if s equals 0.

Lemma 3.3. *Let $f :]1, \infty[\to \mathbb{C}$, $f \in \mathcal{C}^2$ and let $f(\delta(P, Q))$ be a solution of he differential equation*

$$-\Delta f(\delta(P, Q)) = \lambda f(\delta(P, Q)),$$

where Δ acts on $P \in \mathbb{H} \setminus \{Q\}$. Suppose $\lambda \neq 1$. Then $f(\delta)$ is a linear combination of $\varphi_s(\delta)$ and $\varphi_{-s}(\delta)$.

Theorem 3.4. *Let $k :]1, \infty[\to \mathbb{C}$ be a mesurable function and let $f : \mathbb{H} \to \mathbb{C}$ be a solution of the differential equation $-\Delta f = \lambda f$.*

If k and f are non-negative and if the Lebesgue integral

$$h(\lambda) = \frac{\pi}{s} \int_1^\infty k\left(\frac{1}{2}\left(t + \frac{1}{t}\right)\right)(t^s - t^{-s})\left(t - \frac{1}{t}\right)\frac{dt}{t} \tag{1}$$

exists, then the function $k(\delta(P, \cdot))\ f(\cdot)$ is v-integrable over $\mathbb{H}$ for all $P \in \mathbb{H}$, and the equation

$$\int_{\mathbb{H}} k(\delta(P, Q))\ f(Q)\ dv(Q) = h(\lambda)\ f(P) \tag{2}$$

holds.

If $k(\delta(P, \cdot))\ f(\cdot)$ is v-integrable over $\mathbb{H}$, then the Lebesgue integral (1) *exists and* (2) *holds.*

For $s = 0$ the factor $\frac{1}{s}(t^s - t^{-s})$ in (1) *has to be replaced by its limit $2 \log t$. The integral transform* (1) *is called the Selberg transform.*

Proof. To compute $\int_{\mathbb{H}} k(\delta(P, Q))\ f(Q)\ dv(Q)$, we transform $\mathbb{H}$ to $\mathbb{B}$ so that $P \mapsto 0$, $Q \mapsto x \in \mathbb{B}$. Then we obtain a solution $g : \mathbb{B} \to \mathbb{C}$ of $-\Delta g = \lambda g$ such that $f(P) = g(0)$, and g is non-negative whenever f is non-negative. Then by a change of variables and Fubini's theorem we get

$$\int_{\mathbb{H}} k(\delta(P, Q))\ f(Q)\ dv(Q) = \int_{\mathbb{B}} k(\delta'(0, x))\ g(x)\ dv'(x)$$

$$= \int_0^1 k\left(\frac{1}{2}\left(\frac{1+\rho}{1-\rho} + \frac{1-\rho}{1+\rho}\right)\right) \int_{\mathbb{S}^2} g(\rho\zeta)\ d\Omega(\zeta)\ \frac{8\rho^2 d\rho}{(1-\rho^2)^3},$$

where $d\Omega$ denotes the Euclidean surface measure on $\mathbb{S}^2$. By the uniform convergence of g on compact subsets of $\mathbb{B}$ the integral over $\mathbb{S}^2$, the integral over $\mathbb{S}^2$ is computed by termiwise integration. Only the zeroth summand gives a contribution since all integrals of $Y_{\ell j}(\zeta)$ with $\ell > 1$ vanish. Then we obtain

$$\int_{\mathbb{H}} k(\delta(P, Q))\ f(Q)\ dv(Q)$$

$$= \frac{\pi}{s}\int_0^1 k\left(\frac{1}{2}\left(\frac{1+\rho}{1-\rho}+\frac{1-\rho}{1+\rho}\right)\right)\left(\left(\frac{1+\rho}{1-\rho}\right)^s-\left(\frac{1-\rho}{1+\rho}\right)^s\right)\frac{8\rho^2 d\rho}{(1-\rho^2)^2}\, f(P)$$

$$= \frac{\pi}{s}\int_1^\infty k\left(\frac{1}{2}\left(t+\frac{1}{t}\right)\right)(t^s-t^{-s})\left(t-\frac{1}{t}\right)\frac{dt}{t}\, f(P).$$

Provided that the left-hand side is integrable. Both sides of (2) are allowed to be infinite. But if (1) is finite then the left-hand side of (2) is also finite, and (2) holds. □

Theorem 3.5. *Let $f : \mathbb{H} \to \mathbb{C}$ be a $\mathcal{C}^2$-function which is harmonic in tje hyperbolic sense and let $B_R(P)$ denote the hyperbolic ball with center P and radius $R > 0$. Then f satisfies the mean-value formula*

$$f(P) = \frac{1}{v\left(B_R(P)\right)}\int_{B_R(P)} f(Q)\ dv(Q). \tag{3}$$

We define the Schwartz space of a metric space Ω, $\mathcal{S}(\Omega)$ to be the space of $\mathcal{C}^\infty$ functions $f : \Omega \to \mathbb{R}$ which $x^n f^{(\ell)}(x)$ is bounded for every $n \in \mathbb{N}$ as $x \to \infty$. We say that f is of rapid decrease if satisfies the second property.

Lemma 3.6. *Let $k \in \mathcal{S}([1,\infty[)$ then the Selberg transform h of k defined in (1) exists for every $s \in \mathbb{C}$. Define for $x \in \mathbb{R}$*

$$g(x) := 2\int_{\mathbb{R}^2} k\left(u_1^2+u_2^2+\cosh(x)\right)\ du_1\ du_2 = 2\pi\int_0^\infty k(u+\cosh(x))\ du. \tag{4}$$

Then g is even and of rapid decrease as $|x| \to \infty$ and

$$h(1+t^2) = \int_{-\infty}^\infty g(x)\mathrm{e}^{itx}\ dx, \quad g(x) = \frac{1}{2\pi}\int_{-\infty}^\infty h(1+t^2)\mathrm{e}^{-itx}\ dt. \tag{5}$$

We also have the formula:

$$\int_{\mathbb{R}^2} k\left(\frac{\|u\|^2+r_1^2+r_2^2}{2r_1r_2}\right)\ du = r_1r_2 g\left(\log\left(\frac{r_1}{r_2}\right)\right) \quad r_1, r_2 > 0. \tag{6}$$

Define further for $x \geq 1$

$$Q(x) := 2\pi\int_0^\infty k(x+t)\ dt. \tag{7}$$

Then we have

$$g(x) = Q(\cosh(x)), \quad Q'(x) = -2\pi k(x). \tag{8}$$

Proof. Taking $f(P) := r_P^{1+s}$, it is sufficient to consider $P = j$ and $Q = z + jr = z_1 + iz_2 + jr$. Setting $r = e^x$, $s = it$ and $u_m = \frac{z_m}{\sqrt{2r}}$ for $m = 1, 2$. Then by Theorem 3.4

$$h(1+t^2) = \int_{\mathbb{H}} k(\delta(j,Q))r^{1+it}\ dv(Q) = \int_0^\infty\int_{\mathbb{R}^2} k\left(\frac{|z|^2}{2r}+\frac{r^{-1}+r}{2}\right) r^{1+it}\ \frac{dz_1\ dz_2\ dr}{r^3}$$

$$= 2\int_{-\infty}^{\infty}\int_{\mathbb{R}^2} k(u_1^2+u_2^2+\cosh(x))e^{itx}\ du_1\ du_2\ dx = \int_{-\infty}^{\infty} g(x)e^{itx}.$$

Taking the inverse Fourier transform we obtain the other formula. The function g is obviously of rapid decrease since $k \in \mathcal{S}([1,\infty[)$.

$$g\left(\log\frac{r_1}{r_2}\right) = 2\int_{\mathbb{R}^2} k\left(u_1^2+u_2^2+\cosh\left(\log\frac{r_1}{r_2}\right)\right)\ du_1\ du_2$$

$$= \frac{1}{r_1r_2}\int_{\mathbb{R}^2} k\left(\frac{|z|^2+r_1^2+r_2^2}{2r_1r_2}\right)\ dz_1\ dz_2, \quad r_1, r_2 > 0.$$

□

3.2 The Resolvent Kernel

Define the second Sobolev space

$$H^2(\Gamma\setminus\mathbb{H}) := \{f \in L^2(\Gamma\setminus\mathbb{H}) :\ \Delta f \in L^2(\Gamma\setminus\mathbb{H})\}.$$

Definition 3.6. Let $\Gamma < \mathbf{PSL}_2(\mathbb{C})$ be a discrete group. The natural domains of definition of Δ are

$$\mathcal{D} := H^2(\Gamma\setminus\mathbb{H}) \cap \mathcal{C}^2(\mathbb{H}),$$

$$\mathcal{D}^\infty = \{f \in L^2(\Gamma\setminus\mathbb{H}) \cap \mathcal{C}^\infty(\mathbb{H}) :\ \pi_\Gamma(\mathrm{supp}(f)) \text{ is compact in } \Gamma\setminus\mathbb{H}\}$$

where $\pi_\Gamma : \mathbb{H} \to \Gamma\setminus\mathbb{H}$ is the natural projection, and $\mathrm{supp}(f)$ is the support of f.

Then one has the following result.

Theorem 3.7. *Let $\Gamma < \boldsymbol{PSL}_2(\mathbb{C})$ be a discrete group. Then the operators*

$$\Delta : \mathcal{D}^\infty \to L^2(\Gamma\setminus\mathbb{H}), \qquad \Delta : \mathcal{D} \to L^2(\Gamma\setminus\mathbb{H})$$

are essentially self-adjoint and have the same self-adjoint extension. We define

$$\tilde{\Delta} : \tilde{\mathcal{D}} \to L^2(\Gamma\setminus\mathbb{H})$$

to be the unique self-adjoint extension of $\Delta : \mathcal{D}^\infty \to L^2(\Gamma\setminus\mathbb{H})$ and of $\Delta : \mathcal{D} \to L^2(\Gamma\setminus\mathbb{H})$.

Lemma 3.8. *$\mathcal{D}^\infty$ coincides with the set of functions $w : \mathbb{H} \to \mathbb{C}$ having a representation of the form*

$$w = \sum_{M\in\Gamma} h \circ M,$$

where $h \in \mathcal{C}_c^\infty(\mathbb{H})$ ($\mathcal{C}^\infty$-function with compact support).

Lemma 3.9. *Let $t > 1$ and $Q \in \mathbb{H}$. Then there exists a constant $C > 0$ independent of Q such that for all $M \in \boldsymbol{PSL}_2(\mathbb{C})$*

$$\int_{\mathbb{H}} \delta(MP, Q)^{-t} \delta(P, Q)^{-t} \, dv(P) \leq C \, \frac{1 + \log \delta(MQ, Q)}{\delta(MQ, Q)^t}. \tag{9}$$

C can be chosen as a continuous function of t.

Lemma 3.10. *Let $s \in \mathbb{C}$,*

$$\varphi_s(\delta) := \frac{1}{4\pi} \, \frac{(\delta + \sqrt{\delta^2 - 1})^{-s}}{\sqrt{\delta^2 - 1}} \qquad \delta > 1,$$

and suppose $u \in \mathcal{C}_c^2(\mathbb{H})$. Then

$$u(Q) = \int_{\mathbb{H}} \varphi_s(\delta(P, Q)) \, (-\Delta - \lambda) u(P) \, dv(P). \tag{10}$$

Proof. We transform $\mathbb{H}$ onto $\mathbb{B}$ isometrically such that $Q \mapsto 0$ and $P \mapsto x = (\xi, \eta, \zeta)$. Setting $\rho = \|x\|$. On $\mathbb{B}$, we consider the differential form

$$\omega = \frac{2}{1 - \rho^2}((f_\xi g - f g_\xi) d\eta \wedge d\zeta + (f_\eta g - f g_\eta) d\zeta \wedge d\xi + (f_\zeta g - f g_\zeta) d\xi \wedge d\eta).$$

Then we have $d\omega = ((\Delta f) g - f \Delta g)) \, dv'$. We transform the integral to an integral over $\mathbb{B}$ and we exclude a small Euclidean all of radius ε and centre 0. We denote the inverse image of $\mathbb{B}_\varepsilon := \{x \in \mathbb{B} : \ \|x\| > \varepsilon\}$ in $\mathbb{H}$ by $\mathbb{H}_\varepsilon$. Then

$$\int_{\mathbb{H}} \varphi_s(\delta(P, Q)) \, (-\Delta - \lambda) u(P) \, dv(P) = \lim_{\varepsilon \to 0} \int_{\mathbb{H}_\varepsilon} \varphi_s(\delta(P, Q)) \, (-\Delta - \lambda) u(P) \, dv(P),$$

and

$$\int_{\mathbb{H}_\varepsilon} \varphi_s(\delta(P, Q)) \, (-\Delta - \lambda) u(P) \, dv(P)$$
$$= \int_{\mathbb{H}_\varepsilon} (\Delta \varphi_s(\delta(P, Q)) \, u(P) - \varphi_s(\delta(P, Q)) \Delta u(P)) \, dv(P) = \int_{\mathbb{B}_\varepsilon} d\omega,$$

where we need to add

$$f(x) = \frac{1}{4\pi} \frac{1 - \rho^2}{2\rho} \left(\frac{1 - \rho}{1 + \rho} \right)^s = \varphi_s(\delta(P, Q)), \quad g(x) = u(P).$$

Let S_ε be the boundary of $\mathbb{B}_\varepsilon$ with orientation such that the normal is directed outside. Introducing spherical coordinates, we obtain the restriction of the differential form ω to the sphere S_ε from

$$f_\xi d\eta \wedge d\zeta + f_\eta d\zeta \wedge d\xi + f_\zeta d\xi \wedge d\eta = \frac{1}{\rho} \langle (\xi, \eta, \zeta), \mathrm{grad}(f) \rangle \rho^2 d\Omega,$$

where $\rho = \varepsilon$, $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^3$, $\mathrm{grad}(f)$ is the Euclidean gradient of f, and Ω is the Euclidean surface measure on $\mathbb{S}^2$. Hence we obtain

$$\int_{\mathbb{B}_\varepsilon} d\omega = -\int_{S_\varepsilon} \omega = -2(1-\varepsilon^2)^{-1} \int_{\mathbb{S}^2} \frac{1}{\rho} \langle (\xi, \eta, \zeta), \mathrm{grad}(f) \rangle g \ \rho^2 d\Omega$$

$$+2(1-\varepsilon^2)^{-1} \int_{\mathbb{S}^2} \frac{1}{\rho} \langle (\xi, \eta, \zeta), \mathrm{grad}(g) \rangle f \ \rho^2 d\Omega.$$

We have that $f(x) = O(\rho^{-1})$ as x tends to 0. Then the second integral converges to 0 as $\varepsilon \to 0$. In the first integral we have

$$\frac{1}{\rho} \langle (\xi, \eta, \zeta), \mathrm{grad}(f) \rangle = \frac{1}{4\pi} \frac{\partial}{\partial \rho} \frac{1-\rho^2}{2\rho} \left(\frac{1-\rho}{1+\rho} \right)^s = -\frac{1}{8\pi \rho^2} + O\left(\frac{1}{\rho} \right).$$

Hence $\lim\limits_{\varepsilon \to 0} \int_{\mathbb{B}_\varepsilon} d\omega = g(0) = u(Q)$. □

We proceed to prove that the Maass-Selberg series defines a kernel of Carleman type. It is convenient to decompose the kernel into a continuous part and a part with singularities, such that the contribution of the singularities has small support.
We take $\psi \in \mathcal{C}_c^\infty([0, \infty[)$ sich that ψ is decreasing and

$$\psi(t) = \begin{cases} 1 & 0 \le t \le 2 \\ 0 & t > 3 \end{cases}$$

We define

$$g(t) := \frac{1}{4\pi\sqrt{2}} \frac{1}{\sqrt{t-1}} \psi(t) \quad t > 1.$$

Then the function

$$k_s(t) = \varphi_s(t) - g(t)$$

is continuous for $t \ge 1$ and there exists a continuous function of s C_1 such that $|k_s(t)| \le C_1 t^{-1-\Re s}$ for all $t \ge 1$.
Let $\Gamma < \mathbf{PSL}_2(\mathbb{C})$ be a discrete group with fundamental domain $\mathcal{F}$ and let σ_0 be the abscissa of convergence of Γ. Define

$$G(P, Q) := \sum_{M \in \Gamma} g(\delta(P, MQ)) \quad P, Q \in \mathbb{H}, \ P \ne Q \mod \Gamma,$$

$$K(P, Q, s) := \sum_{M \in \Gamma} k_s(\delta(P, MQ)) \quad P, Q \in \mathbb{H}, \ \Re s > \sigma_0,$$

$$|K|(P, Q, s) := \sum_{M \in \Gamma} |k_s(\delta(P, MQ))| \quad P, Q \in \mathbb{H}, \ \Re s > \sigma_0,$$

$$|F|(P, Q, s) := \sum_{M \in \Gamma} |\varphi_s(\delta(P, MQ))| = F(P, Q, \Re s) \quad P, Q \in \mathbb{H}, \ \Re s > 1.$$

Lemma 3.11. *For $\Re s > \max(0, \sigma_0)$, the integral*

$$\int_{\mathcal{F}} (|F|(P,Q,s))^2 \; dv(Q) \tag{11}$$

converges uniformly on compact sets with respect to P, s in the following sense. For any $\varepsilon > 0$ and all compact subsets $\mathcal{K} \subset \mathbb{H}$, $\mathcal{S} \subset \{s : \Re s > \max(0, \sigma_0)\}$ there exists a compact subset $\mathcal{L} \subset \mathcal{F}$ such that

$$\int_{\mathcal{F}\setminus\mathcal{L}} (|F|(P,Q,s))^2 \; dv(Q) < \varepsilon \tag{12}$$

for all $P \in \mathcal{K}$, $s \in \mathcal{S}$. An analogous statement holds for $G(P,Q)$ and $|K|(P,Q,s)$.

Proof. If $\Re s = \sigma > \max(0, \sigma_0)$ and $L \subset \mathcal{F}$ is a measurable subset

$$\int_{\mathcal{F}\setminus L} (|K|(P,Q,s))^2 \; dv(q) = \sum_{M\in\Gamma}\sum_{N\in\Gamma} \int_{N(\mathcal{F}\setminus L)} |k_s(\delta(P,NQ))| \; |k_s(\delta(P,MQ))| \; dv(Q)$$

$$= \sum_{M\in\Gamma} \int_{\mathbb{H}\setminus(\Gamma L)} |k_s(\delta(P,Q))| \; |k_s(\delta(P,MQ))| \; dv(Q).$$

Now let $\mathcal{K} \subset \mathbb{H}$, $\mathcal{S} \subset \{s : \; \Re s > \max(0, \sigma_0)\}$ be compact sets, choose $P_0 \in \mathcal{K}$ and $t := 1 + \min\{\Re s : \; s \in \mathcal{S}\}$. Let $C > 0$ be such that $|k_s(\delta(P,Q))| \le C \; \delta(P,Q)^{-t}$ for all $P \in \mathcal{K}$, $Q \in \mathbb{H}$ and $s \in \mathcal{S}$ (note that this is possible by the estimate for k_s).
For $L = \emptyset$ and for all $s \in \mathcal{S}$, $P \in \mathcal{K}$ we have

$$\int_{\mathcal{F}} (|K|(P,Q,s))^2 \; dv(Q) \le C^2 \sum_{M\in\Gamma} \int_{\mathbb{H}} \delta(P_0,Q)^{-t}\delta(P_0,MQ)^{-t} \; dv(Q)$$

$$\le C^2 C'(H(P_0,P_0,t-1) - H'(P_0,P_0,t-1)) < \infty$$

Where C' is a continuous function on t (from (9)).
For any subset $E \subset \Gamma$ and $s \in \mathcal{S}$, $P \in \mathcal{K}$

$$\int_{\mathcal{F}\setminus L} (|K|(P,Q,s))^2 \; dv(Q) \le C^2 \sum_{M\in E} \int_{\mathbb{H}\setminus(\Gamma L)} \delta(P_0,Q)^{-t}\delta(P_0,MQ)^{-t} \; dv(Q)$$

$$+ C^2 \sum_{M\in\Gamma\setminus E} \int_{\mathbb{H}} \delta(P_0,Q)^{-t}\delta(P_0,MQ)^{-t} \; dv(Q). \tag{13}$$

If $\varepsilon > 0$ is given, que can choose E such that the second term on the right-hand side of (13) is less than $\varepsilon/2$ (this is possible by (9)). We choose now a compact subset $L \subset \mathcal{F}$ such that the first term also is less than $\varepsilon/2$. This proves (12) for $|K|$.

Since G is a finite sum if P ranges over a compact set and Q rangers over a Poincaré normal polyhedron for Γ we have that the corresponding statements hold for G. The statements for $|K|$ and G imply the truth of the lemma for F. □

Lemma 3.12. *Let $P_0 \in \mathbb{H}$, $s_0 \in \mathbb{C}$, $\Re s_0 > \max(0, \sigma_0)$. Then*

$$\lim_{(P,s)\to(P_0,s_0)} \int_{\mathcal{F}} |F(P,Q,s) - F(P_0,Q,s_0)|^2 \; dv(Q) = 0$$

uniformly on every compact subset of $\mathbb{H} \times \{s : \Re s > \max(0, \sigma_0)\}$. A corresponding statement holds for $G(P,Q)$ and $K(P,Q,s)$.

Proof. Let Φ be any one of the kernels $F(P,Q,s)$, $G(P,Q)$, $K(P,Q,s)$. Choose $\varepsilon > 0$ and let $\mathcal{K} \subset \mathbb{H}$, $\mathcal{S} \subset \{s : \Re s > \max(0,\sigma_0)\}$ be compact sets. By Lemma 3.11 we choose a compact subset $\mathcal{L} \subset \mathcal{F}$ such that

$$\int_{\mathcal{F}\setminus\mathcal{L}} |\Phi(P,Q,s) - \Phi(P',Q,s')|^2 \; dv(Q) < \frac{\varepsilon}{2}$$

for all $P, P' \in \mathcal{K}$, $s, s' \in \mathcal{S}$. The corresponding integral over $\mathcal{L}$ will also be less than $\varepsilon/2$, provided that $P, P' \in \mathcal{K}$, $s, s' \in \mathcal{S}$, $d(P,P') < \eta$, $|s - s'| < \eta$, where $\eta > 0$ is sufficiently small. $\square$

By the last two lemmas we conclude the following corollary.

Corollary 3.12.1. *For $\Re s > \max(0, \sigma_0)$, the Maass-Selberg series $F(P,Q,s)$ is a kernel of Carleman type, and it is continuous in the mean square.*

Theorem 3.13. *Let $\Gamma < \boldsymbol{PSL}_2(\mathbb{C})$ be a discrete group with fundamental domain $\mathcal{F}$ with abscissa of convergence σ_0. Suppose $\Re s > \max(0, \sigma_0)$. Then any element $u \in \widetilde{\mathcal{D}}$ can be represented by a continuous function which satisfies*

$$u(P) = \int_{\mathcal{F}} F(P,Q,s) \; (-\widetilde{\Delta} - \lambda) u(Q) \; dv(Q) \qquad P \in \mathbb{H},\ \lambda = 1 - s^2. \tag{14}$$

Proof. For any $u \in \tilde{\mathcal{D}}$ the integral

$$f(P) := \int_{\mathcal{F}} F(P,Q,s)(-\tilde{\Delta} - \lambda)u(Q) \; dv(Q) \quad (P \in \mathbb{H})$$

exists and the right-hand side remains finite if $F(P,Q,s)$ and $(-\tilde{\Delta} - \lambda)u$ are replaced by $|F|(P,Q,s)$ and $|(-\tilde{\Delta} - \lambda)u|$, respectively. By the dominated convergence theorem we have that

$$f(P) = \int_{\mathbb{H}} \varphi_s(\delta(P,Q))(-\tilde{\Delta} - \lambda)u(Q) \; dv(Q).$$

If $u \in \mathcal{D}^\infty$ then u it has the form $u = \sum_{M\in\Gamma} w \circ M$ with some $w \in \mathcal{C}_c^\infty(\mathbb{H})$. Let $h := (-\Delta - \lambda)w \in \mathcal{C}_c^\infty(\mathbb{H})$ then we have that $(-\Delta - \lambda)u = \sum_{M\in\Gamma} h \circ M$ and hence by the monotone convergence theorem

$$\int_{\mathbb{H}} |\varphi_s(P,Q)| \; |(-\Delta - \lambda)u(Q)| \; dv(Q) \leq \int_{\mathbb{H}} |\varphi_s(P,Q)| \sum_{M\in\Gamma} |h(MQ)| \; dv(Q)$$

$$= \int_{\mathbb{H}} |F|(P, Q, s)\ h(Q)\ dv(Q). \tag{15}$$

Since the compact support of h is covered by finitely many Γ-images of a Poincaré normal polyhedron for Γ, the right-hand side of (16) is finite. Then we can apply the dominated convergence theorem in $f(P)$ for our u and we obtain

$$f(P) = \sum_{m \in \Gamma} \int_{\mathbb{H}} \varphi_s(P, Q)(-\Delta - \lambda)(w \circ M)(Q)\ dv(Q) = \sum_{M \in \Gamma} (w \circ M)(P) = u(P).$$

Then we have proved the theorem for all $u \in \mathcal{D}^\infty$.
Now assume that $u \in \tilde{\mathcal{D}}$. Since the operator $\tilde{\Delta} : \tilde{\mathcal{D}} \to L^2(\Gamma \setminus \mathbb{H})$ is the closure of $\Delta : \mathcal{D}^\infty \to L^2(\Gamma \setminus \mathbb{H})$. Hence there exists a sequence $(u_n)_{n \geq 1}$ in $\mathcal{D}^\infty$ such that $u = \lim\limits_{n \to \infty} u_n$, $\tilde{\Delta} u = \lim\limits_{n \to \infty} \Delta u_n$, the limits are taken in the norm sense. The continuity of the inner product implies

$$f(P) = \langle (-\tilde{\Delta} - \lambda)u, \overline{F(P, \cdot, s)} \rangle = \lim_{n \to \infty} \langle (-\tilde{\Delta} - \lambda)u_n, \overline{F(P, \cdot, s)} \rangle$$

$$= \lim_{n \to \infty} \int_{\mathcal{F}} F(P, Q, s)\ (-\Delta - \lambda)u_n(Q)\ dv(Q) = \lim_{n \to \infty} u_n(P) = u(P).$$

This means that the sequence $(u_n)_{n \geq 1}$ converges to f pointwise and to u in the mean square. hence $f = u$ almost everywhere. Since f is continuous by Lemma 3.12 u can be represented by a continuous function on $\mathbb{H}$ which satisfies (15). $\square$

If $\Gamma < \mathbf{PSL}_2(\mathbb{C})$ has finite covolume then $-\tilde{\Delta}$ is a positive self-adjoint operator ($\sigma(-\tilde{\Delta}) \subset [0, -\infty[$).

Corollary 3.13.1. *Let $\Gamma < \boldsymbol{PSL}_2(\mathbb{C})$ be a discrete group with abscissa of convergence σ_0. Suppose that $\lambda \in \mathbb{C} \setminus [1, \infty[$, and assume that $\Re s > \max(0, \sigma_0)$. Then λ belongs to the resolvent set $\rho(-\widetilde{\Delta})$, and for all $f \in L^2(\Gamma \setminus \mathbb{H})$ the resolvent operator is given by*

$$R_\lambda f = \int_{\mathcal{F}} F(\cdot, Q, s)\ f(Q)\ dv(Q). \tag{16}$$

Theorem 3.14. *Let $\Gamma < \boldsymbol{PSL}_2(\mathbb{C})$ be a discrete cofinite group with fundamental domain $\mathcal{F}$, suppose $\lambda, \mu \in \mathbb{C} \setminus [1, \infty[$, $\lambda = 1 - s^2$, $\mu = 1 - t^2$ and assume $\Re s, \Re t > 1$. Then*

$$(\lambda - \mu) \int_{\mathcal{F}} F(P, Z, s) F(Z, Q, t)\ dv(Z) = \lim_{Z \to} (F(P, Z, s) - F(P, Z, t)) \tag{17}$$

for all $P, Q \in \mathbb{H}$. In particular, for non-real λ we have

$$\int_{\mathcal{F}} |F(P, Z, s)|^2\ dv(Z) = \frac{1}{\lambda - \overline{\lambda}} \lim_{z \to P} (F(P, Z, s) - F(P, Z, \overline{s})) \tag{18}$$

and for $\lambda \in \mathbb{R}$ *we have*

$$\int_{\mathcal{F}} |F(P,Z,s)|^2 \; dv(Z) = \lim_{Z \to P} \frac{\partial}{\partial \lambda} F(P,Z,s) \tag{19}$$

$$= \lim_{\mu \to \lambda} \frac{1}{\lambda - \mu} (\lim_{Z \to P} (F(P,Z,s) - F(P,Z,t))).$$

Proof. By Hilbert's resolvent equation

$$(\lambda - \mu) R_\lambda R_\mu = R_\lambda - R_\mu \qquad \lambda, \mu \in \rho(-\tilde{\Delta}).$$

By Corollary 3.12.1 we have that

$$\int_{\mathcal{F}} |F(P,Z,s)||F(Z,Q,t)| \; dv(Z)$$

is a continuous function on $Q \in \mathbb{H}$. Then

$$\int_{\mathcal{F}} \int_{\mathcal{F}} |F(P,Z,s)||F(Z,Q,t)| \; dv(Z) |f(Q)| \; dv(Q)$$

is finite for all $f \in \mathcal{D}^\infty$. We conclude that

$$(\lambda - \mu) \int_{\mathcal{F}} F(P,Z,s) \int_{\mathcal{F}} F(Z,Q,t) \; f(Q) \; dv(Q) \; dv(Z)$$

$$= (\lambda - \mu) \int_{\mathcal{F}} \int_{\mathcal{F}} F(P,Z,s) \; F(Z,Q,t) \; dv(Z) \; f(Q) \; dv(Q)$$

$$= \int_{\mathcal{F}} (F(P,Q,s) - F(P,Q,t)) \; f(Q) \; dv(Q)$$

holds for all $f \in \mathcal{D}^\infty$, $P \in \mathbb{H}$ by Fubini's theorem (both sides are continuous functions of P by Theorem 3.13) and then

$$(\lambda - \mu) \int_{\mathcal{F}} F(P,Z,s) \; F(Z,Q,t) \; dv(Z) = F(P,Q,s) - F(P,Q,t)$$

$$= \lim_{Z \to Q} (F(P,Z,s) - F(P,Q,t)) \quad \forall P \in \mathbb{H}$$

holds for almost all $Q \in \mathbb{H}$ and both sides of the equation are continuous functions of $Q \in \mathbb{H}$. $K(P,Q,s)$ is continuous on $\mathbb{H} \times \mathbb{H} \times \{s : \Re s > 1\}$. Then (18) holds for all $P, Q \in \mathbb{H}$ and the other assertions are true since $\overline{F(P,Q,s)} = F(P,Q,\overline{s})$. □

We say that the resolvent R_λ ($\lambda = 1 - s^2$) is of Hilbert-Schmidt type if its kernel has finite norm, which means

$$\int_{\mathcal{F}} \int_{\mathcal{F}} |F(P,Q,s)|^2 \; dv(Q) \; dv(P) < \infty.$$

Theorem 3.15. *Let $\Gamma < \boldsymbol{PSL}_2(\mathbb{C})$ be a discrete cocompact group and $\lambda \in \rho(\widetilde{\Delta})$. Then the operator $R_\lambda = (-\widetilde{\Delta} - \lambda)^{-1}$ is of Hilbert-Schmidt type.*

Proof. Let $\mathcal{F}$ be a compact fundamental domain for Γ then

$$\int_{\mathcal{F}} |F(P,Q,s)|^2 \; dv(Q)$$

is a continuous function (by Corollary 3.12.1) and hence

$$\int_{\mathcal{F}} \int_{\mathcal{F}} |F(P,Q,s)|^2 \; dv(Q) \; dv(P) < \infty.$$

□

Corollary 3.15.1. *Suppose that $\Gamma < \boldsymbol{PSL}_2(\mathbb{C})$ is a discrete cocompact group. Then the operator $-\Delta : \mathcal{D} \to L^2(\Gamma \setminus \mathbb{H})$ has a complete orthonormal system $(e_n)_{n \in \mathbb{N}}$ of eigenfunctions with corresponding eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq ...$ counted according to their multiplicities and $\sum_{n=1}^{\infty} \lambda_n^{-2} < \infty$.*

Proof. Since $(-\tilde{\Delta} - \lambda)^{-1}$ is for $\lambda < 0$ a self-adjoint operator of Hilbert-Schmidt type we conclude that has a complete orthonormal system of eigenfunctions $(e_n)_{n \geq 0}$ with associated eigenvalues μ_n such that $\sum_{n=0}^{\infty} \mu_n^2 < \infty$. □

Theorem 3.16. *Let $\Gamma < \boldsymbol{PSL}_2(\mathbb{C})$ be a discrete cocompact group. Then its Maass-Selberg series can be continued meromorhically to all of $\mathbb{C}$.*

4 The Selberg Trace Formula

In this chapter we compute the trace of the resolvent of the Laplace-Beltrami operator. We are not dealing with parabolic elements since we compute the trace for cocompact groups $\Gamma < \mathbf{PSL}_2(\mathbb{C})$ only. In the parabolic case more terms appear into the trace (see [7] for the three-dimensional case and [13] for the two-dimensional case) and the spectrum of the Laplace-Beltrami operator shifts in a discrete part and on a continuous part. After computing the trace we give a few applications of the formula. We follow [7].

4.1 Computation of the Trace

Assume that $\Gamma < \mathbf{PSL}_2(\mathbb{C})$ is a discrete cocompact group. Following the notation of chapter 2, let $(e_n)_{n\geq 0}$ be a complete orthonormal system of eigenfunctions of $-\Delta$: $\mathbb{D} \to L^2(\Gamma\setminus\mathbb{H})$. Then we have $-\Delta e_n = \lambda_n e_n$ $\forall n \geq 0$. Given a function $k : [1,\infty[\to \mathbb{C}$, $k \in \mathcal{S}([1,\infty[)$ and $K := k \circ \delta$ we define

$$\tilde{K}f(P) := \int_{\mathbb{H}} K(P,Q)\ f(Q)dv(Q) \quad P \in \mathbb{H}$$

which is an integral operator. If $-\Delta f = \lambda f$ then $\tilde{K}f(P) = h(\lambda)\ f(P)$ where h is the Selberg transform of k (Theorem 3.4). By Poincaré summation process we define

$$K_\Gamma(P,Q) = \sum_{\gamma\in\Gamma} K(P,\gamma Q) \quad P,Q \in \mathbb{H}$$

which is, by construction, Γ-invariant in both variables. For fixed P $K_\Gamma(P,\cdot) \in L^2(\Gamma\setminus\mathbb{H})$. Analogously as above we define an integral operator on Γ-invariant functions

$$\check{K}_\Gamma f(P) := \int_{\mathcal{F}} K_\Gamma(P,Q)\ f(Q)\ dv(Q) \quad P \in \mathbb{H}.$$

If f is a Γ-invariant eigenfunction of $-\Delta$ with eigenvalue λ then $\check{K}_\Gamma f = h(\lambda)\ f$. We also have $\langle K_\Gamma(P,\cdot), e_m\rangle = h(\lambda_m)\overline{e_m(P)}$. Then

$$K_\Gamma(P,Q) = \sum_{m\geq 0} h(\lambda_m)e_m(Q)\overline{e_m(P)}, \quad \|K_\Gamma(P,\cdot)\|^2 = \sum_{m\geq 0} |h(\lambda_m)e_m(P)|^2.$$

We divide K_Γ in three terms $K_\Gamma = K_\Gamma^{\mathrm{id}} + K_\Gamma^{\mathrm{ell}} + K_\Gamma^{\mathrm{lox}}$, where $K_\Gamma^{\mathrm{id}}, K_\Gamma^{\mathrm{ell}}, K_\Gamma^{\mathrm{lox}}$ are sums of K_Γ restricted to the identity element, the elliptic elements and the loxodromic elements in Γ, respectively. Then the first part of the trace reads

$$\sum_{m\geq 0} h(\lambda_m) = \int_{\mathcal{F}} K_\Gamma(P,P)\ dv(P) = \sum_{\gamma\in\Gamma}\int_{\mathcal{F}} K(P,\gamma P)\ dv(P).$$

In this section we will give a more clear expression of the right hand side of the trace. What we want is to compute the trace of $K_\Gamma(P,\cdot)$ and find an expression whithout eigenvalues (explicitly). We proceed calculating separately each of the three terms.

1) We start with the identity term

$$\int_{\mathcal{F}} K_\Gamma^{\mathrm{id}}(P,P)\ dv(P) = \int_{\mathcal{F}} K(P,P)\ dv(P) = \int_{\mathcal{F}} k(\delta(P,P))\ dv(P)$$

$$= \int_{\mathcal{F}} k(1) dv(P) = k(1)\mathrm{vol}(\Gamma).$$

By Lemma 3.6 we have $k(x) = -\frac{Q'(x)}{2\pi}$ and $g'(x) = Q'(\cosh(x))\sinh(x)$ and since

$$g(x) = \frac{1}{2\pi}\int_{\mathbb{R}} h(1+t^2)e^{-itx}dt = \frac{1}{2\pi}\int_{\mathbb{R}} h(1+t^2)\cos(tx)\ dt$$

we obtain

$$k(\cosh(x)) = \frac{Q'(\cosh(x))}{2\pi} = \frac{g'(x)}{2\pi\sinh(x)} = \frac{1}{4\pi^2\sinh(x)}\int_{\mathbb{R}} h(1+t^2)t\sin(tx)\ dt \quad (20)$$

$$\Longrightarrow k(1) = \lim_{x\to 0}\frac{1}{4\pi^2\sinh(x)}\int_{\mathbb{R}} h(1+t^2)t\sin(tx)\ dt =$$

$$\frac{1}{4\pi^2}\int_{\mathbb{R}} h(1+t^2)t\lim_{x\to 0}\frac{\sin(tx)}{\sinh(x)}\ dt = \frac{1}{4\pi^2}\int_{\mathbb{R}} h(1+t^2)\ t^2\ dt$$

$$\Longrightarrow \int_{\mathcal{F}} K_\Gamma^{id}(P,P)\ dv(P) = \frac{\mathrm{vol}(\Gamma)}{4\pi^2}\int_{\mathbb{R}} h(1+t^2)\ t^2\ dt.$$

The integral is convergent and we can introduce the limit into the integral because $k \in \mathcal{S}([1,\infty[)$. We can relax this condition as we will show. First, suppose that $h(1+z^2) = O((1+|z|^2)^{-n})$ that is $h(1+x^2) \leq C(1+x^2)^{-n}$, $x \in \mathbb{R}$ for some $n \in \mathbb{R}$ and for some constant $C > 0$. Taking $f(x,t) := h(1+x^2)x\frac{\sin(tx)}{\sinh t}$, $x \in \mathbb{R}$ and $G(x) = C(1+x^2)^{-n}x^2$ then there exist $\delta > 0$ such that for all $t \in [-\delta,\delta]$ we have

$$\left|\frac{\sin(tx)}{\sinh t}\right| \leq \left|\frac{\sin(tx)}{t}\right| \leq |x|,$$

$$|f(x,t)| \leq C(1+x^2)^{-n}x^2 = G(x),\quad \forall t \in [-\delta,\delta],\quad \forall x \in \mathbb{R}$$

$$\int_{\mathbb{R}} G(x)\ dx < \infty \iff n < -3/2.$$

So by the dominated convergence theorem we conclude that $f(x,t)$ satisfies the desired hypotesis to interchange the limit with the integral if $n = -3/2 - \varepsilon$ for some $\varepsilon > 0$.

Suppose that $h(1+z^2)$ is analytic in a strip $|\Im z| \leq \sigma$ for $\sigma = 3/2 + \varepsilon$ (for some $\varepsilon > 0$).

Step I : $h(1+z^2) = O(e^{-c(\Re z)^2})$ for some $c > 0$. Then, since $(-it)^\nu h(1+t^2)e^{-itx}$ is an integrable function (for all $\nu \geq 0$) we can interchange the derivative with the integral to obtain

$$g^{(\nu)}(x) = \int_{\mathbb{R}} (-it)^\nu h(1+t^2)e^{-itx}\, dt,$$

the analicity of h gives (for some $M, M' > 0$)

$$|g^{(\nu)}(x)| \leq \int_{\mathbb{R}} |(-it)^\nu h(1+t^2)e^{-itx}|\, dt = M \int_{\mathbb{R}-i\sigma} |t^\nu e^{-ct^2} e^{-itx}|\, dt$$

$$= Me^{-\sigma|x|} \int_{\mathbb{R}} |(t+i\sigma)^\nu e^{-ct^2-+c\sigma^2}|\, dt \leq M' e^{-\sigma|x|} \implies g^{(\nu)}(x) = O(e^{-\sigma|x|}).$$

Now, let φ be any $\mathcal{C}^\infty$ function such that:
a) $0 \leq \varphi(x) \leq 1$,
b) $\varphi(x) = 1$ for $0 \leq x \leq 1$,
c) $\varphi(x) = 0$ for $2 \leq x \leq \infty$,
d) φ monotonic decreasing on $\mathbb{R}_+$.
Define

$$\varphi_m(x) = \begin{cases} 1 & |x| \leq m \\ \varphi(|x|-m) & |x| > m \end{cases}$$

Then $\varphi_m \in \mathcal{C}^\infty(\mathbb{R})$, φ_m is even and $\varphi_m(x) = 0$ for $|x| \leq m+2$. We define $g_m(x) = g(x)\varphi_m(x) \in \mathcal{C}_c^\infty(\mathbb{R})$ and $Q_m(\cosh(x)) = g_m(x)$. We also have $g_m^{(\nu)}(x) = O(e^{-\sigma|x|})$ uniformly on m and x for all $\nu \geq 0$. Defining $h_m(1+t^2)$ as the transform of $g_m(x)$ we obtain for $|\Im t| \leq 3/2$

$$|h_m(1+t^2)| \leq \int_{\mathbb{R}} |g_m(x)e^{\Im tx}|\, dx \leq M'' \left| \int_0^{m+2} e^{-\sigma x}(e^{-x\Im t} + e^{x\Im t})\, dx \right|$$

$$\leq \frac{2}{|(\Im t)^2 - \sigma^2|} + \frac{1}{|\Im t - \sigma|} e^{(m+2)(\Im t - \sigma)} + \frac{1}{|\Im t + \sigma|} e^{-(m+2)(\Im t + \sigma)}.$$

The dominated convergence theorem leads us to introduce the limit into the integral

$$\lim_{m\to\infty} h_m(1+z^2) = h(1+z^2) \quad |\Im z| \leq 3/2.$$

Step II : Suppose now that $h(1+z^2) = O((1+|z|^2)^{-\sigma})$. Consider $h_\epsilon(1+z^2) = h(1+z^2)e^{-\epsilon z^2}$ for $\epsilon > 0$, $|\Im z| \leq \sigma$. Then

$$|h_\epsilon(1+z^2)| = |h(1+z^2)|e^{\epsilon((\Im z)^2 - (\Re z)^2)} \leq M(\varepsilon)M(1+|z|^2)^{-\sigma}$$

for $|\Im z| \leq \sigma$. Thus, $h_\epsilon(1+z^2)$ works in step I for each $\epsilon > 0$. Then

$$g(x) = \frac{1}{2\pi} \int_{\mathbb{R}} h(1+t^2)e^{-itx}\, dt = \frac{1}{2\pi} \int_{\mathbb{R}-i\sigma} h(1+t^2)e^{-itx}\, dt$$

$$= \frac{1}{\pi} \int_{\mathbb{R}} h(1 + (t + i\sigma)^2) e^{-i(t+i\sigma)x} \, dt$$

$$\implies |g(x)| \leq \frac{1}{2\pi} \int_{\mathbb{R}} |h(1 + (t + i\sigma)^2)| e^{-\sigma x} \, dt \leq \frac{1}{2\pi} \left(2 - \frac{1}{2^{1-\sigma}(1-\sigma)} \right) MM(\varepsilon) e^{-\sigma x}$$

which gives a bound for $x > 0$. Shifting the transform to $\mathbb{R} + i\sigma$ we obtain the same bound for $x < 0$ and then

$$|g(x)| \leq \frac{1}{2\pi} \left(2 - \frac{1}{2^{1-\sigma}(1-\sigma)} \right) MM(\varepsilon) e^{-\sigma|x|} \quad \forall x \in \mathbb{R}.$$

The same argument applies for $g_\epsilon(x) = \frac{1}{2\pi} \int_{\mathbb{R}} h_\epsilon(1 + t^2) e^{-itx} \, dt$. Finally, taking $\epsilon \to 0$ all the properties (these properties can be applied to the other terms K_Γ^{ell} K_Γ^{lox} to obtain similar results) hold and then we require $h(1 + z^2)$ to be analytic in the strip $|\Im z| \leq \sigma$ for $\sigma = 3/2 + \varepsilon$ (for $\varepsilon > 0$) and satisfying $h(1+z^2) = O((1+|z|^2)^\sigma)$ uniformly in the strip.

Now we proceed to calculate the terms invlolving elliptic and loxodromic elements following Selberg's method ([26]). Let $\{T\} = \{T\}_\Gamma$ run through the set of Γ-conjugacy classes of the elements $M \in \Gamma$ $M \neq \mathbb{1}$ (note that in Selberg's method one takes primitive classes). Then one observes that $S^{-1}TS = M$ runs through $\{T\}$ exactly once whenever S runs through a system $\mathcal{C}(T) \setminus \Gamma$ of representatives of the right cosets of the centralizer $\mathcal{C}(T)$ of T in Γ

$$M = S^{-1}TS = L^{-1}TL \iff \left(SL^{-1}\right)^{-1} T \left(SL^{-1}\right) = T \iff SL^{-1} \in \mathcal{C}(T)$$

for $S, L \in \Gamma$ that is if SL^{-1} is in the same class of $\mathbb{1}$ and then is represented by $\mathbb{1}$ in $\mathcal{C}(T) \setminus \Gamma$. This helps us to simplify the integral

$$\sum_{M \in \Gamma,\ M \neq \mathbb{1}} \int_{\mathcal{F}} K(P, MP) \, dv(P) = \sum_{\{T\}} \sum_{M \in \{T\}} \int_{\mathcal{F}} K(P, MP) \, dv(P)$$

$$= \sum_{\{T\}} \sum_{S \in \mathcal{C}(T) \setminus \Gamma} \int_{\mathcal{F}} K(P, S^{-1}TSP) \, dv(P) = \sum_{\{T\}} \sum_{S \in \mathcal{C}(T) \setminus \Gamma} \int_{S\mathcal{F}} K(Q, TQ) \, dv(Q)$$

$$= \sum_{\{T\}} \int_{\mathcal{F}(\mathcal{C}(T))} K(Q, TQ) \, dv(Q),$$

where $\mathcal{F}(\mathcal{C}(T))$ denotes the fundamental domain of $\mathcal{C}(T)$. Now we will compute the series by two steps, the first case when T is hyperbolic or loxodromic and the second case when T is elliptic.

2) Suppose that $T \in \Gamma$ is hyperbolic or loxodromic. Then T is conjugate in $\mathbf{PSL}_2(\mathbb{C})$ to a unique element

$$D(T) = \begin{pmatrix} a(T) & 0 \\ 0 & a(T)^{-1} \end{pmatrix}$$

such that $|a(T)| > 1$. We have $D(T)(z+rj) = K(T)z+N(T)rj$, where $K(T) = a(T)^2$ is the *mulitplier* of T and where $N(T) = |a(T)|^2 = |K(T)| > 1$ is the *norm* of T. Since we want to find the structure of $\mathcal{C}(T)$ we need to know which elements of Γ commute with T. Choosing $V \in \mathbf{PSL}_2(\mathbb{C})$ such that $T = V^{-1}D(T)V$ then if $M = V^{-1}NV$ commutes with T

$$\mathbb{1} = MTM^{-1} = V^{-1}ND(T)N^{-1}V \iff \mathbb{1} = ND(T)N^{-1}$$

which is equivalent to N being a diagonal matrix and N being the normal form of M ($N = D(M)$). This is also equivalent to M having the same fixed points in $\mathbb{C} \cup \{\infty\}$ as T. Let $\mathcal{E}(T)$ be the set of elements of finite order contained in $\mathcal{C}(T)$. Then $\mathcal{C}(T)$ is the direct product of $\mathcal{E}(T)$ with an infinite cyclic group $\langle T_0 \rangle$ generated by some hyperbolic or loxodromic element $T_0 \in \mathcal{C}(T)$. This direct product decomposition is preserved if T_0 is replaced by T_0E or T_0^{-1} for some $E \in \mathcal{E}(T)$, note that they have the same norm (since E has finite order we conclude that its norm must be 1). Then we have that T_0 is not uniquely determined by T but yes it is the norm $N(T_0)$. $N(T_0)$ is the minimal norm of a hyperbolic or loxodromic element from $\mathcal{C}(T)$. T_0 is called ([26]) a *primitive hyperbolic or loxodromic element for T in Γ*, respectively. We can impose the normalization condition $T = T_0^k E$ for some $k \geq 1$ and $E \in \mathcal{E}(T)$ on T_0.

It's easy to see that the elements of $\mathcal{E}(T)$ are hyperbolic rotations around the axis of T and that is a cyclic group generated by a rotation with rotation angle $\frac{2\pi}{|\mathcal{E}(T)|}$. Taking V as above we have a simple expression of the fundamental domain of $V\mathcal{C}(T)V^{-1}$

$$\mathcal{F}_0 = \left\{ \rho e^{i\varphi} + rj \ : \ \rho \geq 0,\ 0 \leq \varphi \leq \frac{2\pi}{|\mathcal{E}(T)|},\ 1 \leq r \leq N(T_0) \right\}.$$

Since we get a simple expression for $\mathcal{F}(\mathcal{C}(T))$ (by conjugation) we are now able to simplify the computation of the integral

$$\begin{aligned}
\int_{\mathcal{F}(\mathcal{C}(T))} K(Q,TQ)\ dv(Q) &= \int_{\mathcal{F}_0} K(Q,D(T)Q)\ dv(Q) \\
&= \int_{\mathcal{F}_0} k\left(\frac{|K(T)-1|^2|z|^2 + (N(T)^2+1)r^2}{2N(T)r^2}\right)\ \frac{dx\ dy\ dr}{r^3} \\
= \frac{2\pi}{|\mathcal{E}(T)|} \int_1^{N(T_0)} \int_0^\infty & k\left(\frac{|a(T)-a(T)^{-1}|^2}{2}\frac{\rho^2}{r^2} + \frac{N(T)+N(T)^{-1}}{2}\right)\ \rho\ \frac{d\rho\ dr}{r^3} \\
&= \frac{2\pi}{|\mathcal{E}(T)||a(T)-a(T)^{-1}|^2} \int_1^{N(T_0)} \int_{\mathbb{R}} k(u+\cosh(\log N(T)))\ \frac{1}{r}\ du\ dr \\
= \frac{2\pi \log N(T_0)}{|\mathcal{E}(T)||a(T)-a(T)^{-1}|^2} & \int_{\cosh(\log N(T))}^\infty k(u)\ du = \frac{g(\log N(T)) \log N(T_0)}{4|\mathcal{E}(T)||\sinh(\log a(T))|^2}.
\end{aligned}$$

3) Take now an elliptic element $R \in \Gamma$. Then R is a rotation around a hyperbolic line which remain pointwise fixed by R and which meets $\mathbb{C} \cup \{\infty\}$ in the fixed points

of R in $\mathbb{C} \cup \{\infty\}$. Now we take a rotation R_0 with minimal rotation angle of the subgroup of Γ containing all the elements of Γ with the same fixed points in $\mathbb{C} \cup \{\infty\}$ as R. Then $\exists k : \ R = R_0^k \quad 1 \le k < \mathrm{ord}(R_0)$. We have that $\langle R_0 \rangle$ is a subgroup of $\mathcal{C}(R)$. R_0 is called a *primitive elliptic element of Γ associated with R*. R_0 is uniquely determined up to inversion $k \mapsto \mathrm{ord}(R_0) - k$.

We claim that $\mathcal{C}(R) \neq \langle R_0 \rangle$. Choose $V \in \mathbf{PSL}_2(\mathbb{C})$ such that

$$R_0 = V^{-1} R \left(\frac{2\pi}{m} \right) V,$$

where

$$R(\varphi) := \begin{pmatrix} e^{\frac{i\varphi}{2}} & 0 \\ 0 & e^{-\frac{i\varphi}{2}} \end{pmatrix} \quad \varphi \in \mathbb{R},$$

and $m = \mathrm{ord}(R_0)$. Note that $R(\varphi)$ acts on $\mathbb{H}$ as follows:

$$R(\varphi)(z + rj) = e^{i\varphi} z + rj \quad z \in \mathbb{C}, r > 0$$

and that $a(R) = e^{\frac{i\varphi}{2}}$, $K(R) = a(R)^2 = e^{i\varphi}$, $N(R) = |a(R)|^2 = 1$. If we suppose that $\mathcal{C}(R) = \langle R_0 \rangle$ then, $V\mathcal{C}(R)V^{-1} = \langle R\left(\frac{2\pi}{m}\right) \rangle$ with the following fundamental domain

$$\mathcal{F}_1 = \left\{ \rho e^{i\varphi} + rj \ : \ r > 0, \ \rho \ge 0, \ 0 \le \varphi \le \frac{2\pi}{m} \right\}$$

and then

$$\int_{\mathcal{F}(\mathcal{C}(R))} K(Q, RQ) \ dv(Q) = \int_{V\mathcal{F}(\mathcal{C}(R))} K(V^{-1}Q, RV^{-1}Q)) \ dv(Q)$$

$$= \int_{\mathcal{F}_1} K\left(Q, R\left(\frac{2\pi}{m}\right) Q\right) \ dv(Q) = \int_{\mathcal{F}_1} k \left(\frac{|e^{\frac{2\pi i k}{m}} - 1|^2 |z|^2 + 2r^2}{2r^2} \right) \ \frac{dx \ dy \ dr}{r^3}$$

$$= \int_{\mathcal{F}_1} k \left(1 + 2|z|^2 \left(\sin \frac{\pi k}{m} \right)^2 \right) \frac{dx \ dy \ dr}{r},$$

which diverges for example for $k = \varphi_s - \varphi_0, \quad s \in \mathbb{C}$ (which contradicts the Corollary 3.15.1 since the series $\sum_{m=0}^{\infty} h(\lambda_m)$ is convergent) and since the group structure does not depend on k this is not possible. Then the group $\mathcal{C}(R)$ cannot be finite because the integral remains divergent (since the integrand is invariant under $\mathcal{C}(R)$ and since $\langle R_0 \rangle$ has finite index in $\mathcal{C}(R)$) and hence $\mathcal{C}(R)$ contains a hyperbolic or loxodromic element. There are two cases.

First case : If all elements of $\mathcal{C}(R) \setminus \langle R_0 \rangle$ are hyperbolic or loxodromic then $\langle R_0 \rangle$ coincides with $\mathcal{E}(R)$, the subgroup of $\mathcal{C}(R)$ of all elements of finite order. Every hyperbolic or loxodromic element of $\mathcal{C}(R)$ commutes with R and hence has the same fixed points in $\mathbb{C} \cup \{\infty\}$ as R. Then the fixed line of R is the axis of every hyperbolic or loxodromic element of $\mathcal{C}(R)$. We choose $T_0 \in \mathcal{C}(R) \setminus \mathcal{E}(R)$ such that $N(T_0)$ is minimal. Then we have the direct product decomposition

$$\mathcal{C}(R) = \langle T_0 \rangle \times \mathcal{E}(R),$$

which is the same group structure as in the case of the centralizer of a hyperbolic or loxodromic element. $\mathcal{C}(R)$ is an abelian group. Then the following set is a fundamental domain of $V\mathcal{C}(R)V^{-1}$ in this case

$$\mathcal{F}_1 := \left\{ \rho e^{i\varphi} + rj \;:\; \rho \geq 0,\ 0 \leq \varphi \leq \frac{2\pi}{m},\ 1 \leq r \leq N(T_0) \right\}.$$

Which leads us to simplify the integral

$$\int_{\mathcal{F}(\mathcal{C}(R))} K(P, RQ)\, dv(Q) = \int_{\mathcal{F}_1} k\left(\frac{|K(R)-1|^2|z|^2 + (N(R)^2+1)r^2}{2N(R)r^2} \right) \frac{dx\ dy\ dr}{r^3}$$

$$= \frac{g(\log N(R)) \log N(T_0)}{m|\sin\left(\frac{\pi k}{m}\right)|^2} = \frac{g(\log N(R)) \log N(T_0)}{4|\mathcal{E}(R)||\sinh(\log a(R))|^2}.$$

Here to compute the integral one can proceed as in the step 2). Note that $g(\log N(R)) = g(0)$ and that $a(R) - a(R)^{-1} = 2\sin(\pi/2)$. Here we have written $m = \mathcal{E}(R)$, where $\mathcal{E}(R)$ is the maximal finite subgroup of $\mathcal{C}(R)$. In this case we also have $\mathcal{E}(R) = m = \mathrm{ord}(R)$.

Second case : There exists another element $S \in \mathcal{C}(R) \setminus \langle R_0 \rangle$. Transform R to normal form $R(\varphi)$. Then for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{PSL}_2(\mathbb{C})$ commuting with R is equivalent to have the same fixed points in $\mathbb{C} \cup \{\infty\}$ or if $a = d = 0$ then $\varphi \in \pi + 2\pi\mathbb{Z}$, $e^{\frac{i\varphi}{2}} = \pm i$. Since $S \notin \langle R_0 \rangle$ the second case only is possible. Then R is a hyperbolic rotation with rotation angle π (a half-turn), and $S \neq R$ is an elliptic element of order 2 commuting with R. Let $V \in \mathbf{PSL}_2(\mathbb{C})$ such that $VRV^{-1} = R(\pi)$ then

$$VSV^{-1} = \begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix} \quad b \in \mathbb{C}.$$

If $R_0 \in \mathit{\Gamma}$ is a primitive elliptic element associated with R then $S^{-1}R_0S = R_0^{-1}$ and

$$\mathcal{E}(R) := \langle R_0, S |\ R_0S = SR_0^{-1},\ R_0^m = S^2 = 1 \rangle$$

is a finite subgroup of $\mathcal{C}(R)$ of dihedral type and $\mathcal{E}(R) = 2\mathrm{ord}(R_0)$. The group $\{\mathbb{1}, R, S, RS\}$ is a subgroup of $\mathcal{E}(R)$ isomorphic to the Klein Vierergruppe.
Now we want to see what happens to the other elements of $\mathcal{C}(R) \setminus \mathcal{E}(R)$. Suppose that

$\mathcal{G} < \mathcal{C}(R)$ is a subgroup containing only elements of finite order such that $\mathcal{E}(R) < \mathcal{G}$. Let $A \in \mathcal{G}$, $A \notin \langle R_0 \rangle$. Then it follows form the normal form of R and S that

$$VAV^{-1} = \begin{pmatrix} 0 & \beta \\ -\beta^{-1} & 0 \end{pmatrix} \in \mathbf{PSL}_2(\mathbb{C})$$

for some $\beta \neq 0$. Hence $VASV^{-1} = VAV^{-1}VSV^{-1}$ is a diagonal matrix, and since $AS \in \mathcal{G}$ $(A, S \in \mathcal{G})$ is an element of finite order having the same fixed points in $\mathbb{C} \cup \{\infty\}$ as R, one has $AS \in \langle R_0 \rangle$ and then $\mathcal{G} \subset \mathcal{E}(R)$. Then we proved that $\mathcal{E}(R)$ is a maximal subgroup. A discrete subgroup of $\mathbf{PSL}_2(\mathbb{C})$ contains only elements of finite order if and only if it is finite ([1]).

As in the first case we have that $\mathcal{C}(R)$ contains a hyperbolic or loxodromic element and we know that every hyperbolic or loxodromic element of $\mathcal{C}(R)$ has as axis the fixed line of R. Note that since $R_0 S = S R_0^{-1}$ the elements of $\langle R_o \rangle\ S$ map the fixed line of R onto itself interchanging the endpoints. If $X \in \mathcal{C}(R)$ is hyperbolic or loxodromic, we have

$$AXA^{-1} = X \quad \forall A \in \langle R_0 \rangle,$$
$$BXB^{-1} = X^{-1} \quad \forall B \in \langle R_0 \rangle\ S.$$

Choosing a hyperbolic or loxodromic element $T_0 \in \mathcal{C}(R)$ such that $N(T_0)$ is minimal we have

$$\mathcal{C}(R) = \{T_0^n E : \ E \in \mathcal{E}(R),\ n \in \mathbb{Z}\}.$$

Taking first $T \in \mathcal{C}(R)$ elliptic and $T \notin \langle R_0 \rangle$

$$VTV^{-1} = \begin{pmatrix} 0 & \alpha \\ -\alpha^{-1} & 0 \end{pmatrix} \in \mathbf{PSL}_2(\mathbb{C})$$

has a diagonal matrix when is right-multiplied by VSV^{-1}. Then $VTSV^{-1}$ is a diagonal matrix and $TS\mathcal{C}(R)$ has the same fixed points in $\mathbb{C} \cup \{\infty\}$ as R. Then $TS = T_0^n R_0^\nu$ for some integers, $0 \leq \nu < \mathrm{ord}(R_0), n$ which implies that T has the expected form. If $T \in \mathcal{C}(R)$ is hyperbolic or loxodromic then T has the same fixed points as T_0 and hence there exists $n \in \mathbb{Z}$ such that $E := T_0^{-n} T$ is of finite order. Since E has the same fixed points in $\mathbb{C} \cup \{\infty\}$ as R we have $E \in \langle R_0 \rangle$. In this case have that $\langle T_0, R_0 \rangle$ is an abelian group of index 2 in $\mathcal{C}(R)$ (with $\{\mathbb{1}, S\}$ as a representative system of cosets).

Now we prove that all maximal finite subgroups of $\mathcal{C}(R)$ are conjugate in $\mathbf{PSL}_2(\mathbb{C})$. Let $\mathcal{G}$ be a maximal finite subgroup of $\mathcal{C}(R)$. As before there exists an elliptic element $A \in \mathcal{C}(R)$ of order two such that R and A have different pairs of ficed points in $\mathbb{C} \cup \{\infty\}$ and such that $\mathcal{G} = \langle R_0, A \rangle$. Then $A = T_0^n R_0^\nu S$ for some $n, \nu \in \mathbb{Z}$. Taking $T_1 \in \mathbf{PSL}_2(\mathbb{C})$ a hyperbolic or loxodromic element such that $T_0 = T_1^2$. THen T_1 commutes with R_0, and $T_1^{-n} R_0^\ell A T_1^n = R_0^{\ell+\nu} S \ \ \forall \ell \in \mathbb{Z}$. Then $\mathcal{E}(R) = T_1^{-n} \mathcal{G} T_1^n$ which means that $\mathcal{G}$ is conjugate to $\mathcal{E}(R)$. All results are summarized in the following theorem.

Theorem 4.1. *Let $\Gamma < \boldsymbol{PSL}_2(\mathbb{C})$ be a discrete cocompact group. Suppose that $R \in \Gamma$ is elliptic, and let $R_0 \in \Gamma$ be a primitive elliptic transformation associated with R. Then the centralizer $\mathcal{C}(R)$ of R in Γ contains hyperbolic or loxodromic elements. There are two possibilities. Either all the elliptic elements of $\mathcal{C}(R)$ are contained in $\mathcal{E}(R) := \langle R_0 \rangle$, the cyclic group generated by R_0. Then $\mathcal{C}(R)$ is abelian, and*

$$\mathcal{C}(R) = \langle T_0, R_0 \rangle, \tag{21}$$

where $T_0 \in \mathcal{C}(R)$ is a hyperbolic or loxodromic element such that the norm $N(T_0)$ is minimal. In this case, all the elements in $\mathcal{C}(R) \setminus \mathbb{1}$ haver the same pair of fixed points in $\mathbb{C} \cup \{\infty\}$.

Or R is elliptic of order 2, and there exists an elliptic element $S \in \mathcal{C}(R)$ also of order 2 whose fixed line meets the fixed line of R orthogonally in a common point. Then for every such S,

$$S^{-1} R_0 S = R_0^{-1}, \tag{22}$$

and

$$\mathcal{E}(R) := \langle R_0, S \rangle \tag{23}$$

is a maximal finite subgroup of $\mathcal{C}(R)$. $\mathcal{E}(R)$ is of dihedral type. All the maximal finite subgroups of $\mathcal{C}(R)$ are conjugate in $\boldsymbol{PSL}_2(\mathbb{C})$. The centralizer $\mathcal{C}(R)$ has the presentation

$$\mathcal{C}(R) = \langle R_0, T_0, S | R_0 T_0 = T_0 R_0,\ S R_0 S^{-1} = R_0^{-1}, S T_0 S^{-1} = T_0^{-1},\ R_0^m = S^2 = 1 \rangle \tag{24}$$

where $T_0 \in \mathcal{C}(R)$ is a hyperbolic or loxodromic element such that the norm $N(T_0)$ is minimal. The group $\langle T_0 \rangle \times \langle R_0 \rangle$ is an abelian subgroup of index 2 in $\mathcal{C}(R)$, and $\{\mathbb{1}, S\}$ constitutes a representative system of the cosets of $\langle T_0 \rangle \times \langle R_0 \rangle$ in $\mathcal{C}(R)$. $\square$.

In this second case we have that $\mathcal{F}_1$ is a fundamental domain for $V \langle T_0, R_0 \rangle V^{-1}$ setting $m = \mathrm{ord}(R_0)$. Since $K(Q, RQ)$ is an invariant function under $\mathcal{C}(R)$ (seen as a function of $Q \in \mathbb{H}$)

$$\int_{\mathcal{F}(\mathcal{C}(R))} K(Q, RQ)\ dv(Q) = \frac{1}{2} \int_{\mathcal{F}_1} K(Q, VRV^{-1}Q)\ dv(Q)$$

Since m is an even integer we can take $k = \frac{m}{2}$ ($R = R_0^k$ is of order 2). We have $|\mathcal{E}(R)| = 2m$ and then

$$\int_{\mathcal{F}(\mathcal{C}(R))} K(Q, RQ)\ dv(Q) = \frac{g(\log N(R)) \log N(T_0)}{4 |\mathcal{E}(R)| |\sinh(\log a(R))|^2}.$$

If we collect all terms in 1), 2) and 3) (the sums taken over the identity, the hyperbolic or loxodromic and over the elliptic respectively) we can see that we have proved the Selberg trace formula for discrete cocompact groups.

Theorem 4.2. *Suppose that $\Gamma < \boldsymbol{PSL}_2(\mathbb{C})$ is a disrete group with compact fundamental domain $\mathcal{F}$, and let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$ be the eigenvalues of $-\Delta : \mathbb{D} \to L^2(\Gamma \setminus \mathbb{H})$. Put $\lambda_n = 1 - s_n^2$ with $s_n \in \mathbb{C}$. Let h be a function such that $h(1+z^2)$ is analytic in the strip $|\Im z| \leq \sigma$ for $\sigma = 3/2 + \varepsilon$ (for $\varepsilon > 0$) and satisfying $h(1+z^2) = O((1+|z|^2)^\sigma)$ as $|z| \to \infty$ uniformly in the strip. Then the Selberg trace formula reads*

$$\sum_{m\in\mathbb{N}} h(\lambda_m) = \frac{\mathrm{vol}(\Gamma)}{4\pi^2}\int_{\mathbb{R}} h(1+t^2)\ t^2\ dt + \sum_{L\in\{T\}\mathrm{lox},\{R\}\mathrm{ell}} \frac{g(\log N(L))\log N(T_0)}{4|\mathcal{E}(L)||\sinh(\log a(L))|^2}. \quad (25)$$

Where the left hand side of the formula is absolutely convergent and where the sum of the right hand side is extended over the Γ-conjugacy classes of hyperbolic or loxodromic elements $T \in \Gamma$ and of elliptic elements $R \in \Gamma$. In a Γ-conjugacy class $L \in \Gamma$ (L elliptic, or hyperbolic or loxodromic) $N(T_0)$ is the minimal norm of a hyperbolic or loxodromic element of $\mathcal{C}(L)$. $\square$.

We note that in [7] page 297 there is an errata in Theorem 5.1. It appears a (as I've seen) wrong π multiplying on the function g in the terms involving elliptic and hyperbolic or loxodromic elements not stabilizing a cusp.

Corollary 4.2.1. *Suppose that $\Gamma < \boldsymbol{PSL}_2(\mathbb{C})$ is a disrete group with compact fundamental domain $\mathcal{F}$, and let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$ be the eigenvalues of $-\Delta : \mathbb{D} \to L^2(\Gamma \setminus \mathbb{H})$. Put $\lambda_n = 1 - s_n^2$ with $s_n \in \mathbb{C}$. Assume that $\lambda = 1 - s^2$, $\mu = 1 - t^2$, with $\Re s$, $\Re t > 1$. Then*

$$(\lambda - \mu)\ \mathrm{tr}(R_\lambda R_\mu) = \sum_{n=0}^{\infty}\left(\frac{1}{s^2 - s_n^2} - \frac{1}{t^2 - s_n^2}\right) \quad (26)$$

$$= -\frac{v(\mathcal{F})}{4\pi}(s-t) + \left(\frac{1}{2s} - \frac{1}{2t}\right)\sum_{\{R\}\ \mathrm{ell}} \frac{\log N(T_0)}{4|\mathcal{E}(R)|(\sin(\frac{\pi k}{m(R)}))}$$

$$+ \sum_{\{T\}\ \mathrm{lox}} \frac{\log N(T_0)}{4|\mathcal{E}(T)||\sinh(\log a(T))|^2}\left(\frac{N(T)^{-s}}{2s} - \frac{N(T)^{-t}}{2t}\right).$$

Proof. First, since $\lambda, \mu \in \rho(\tilde{\Delta})$ we have

$$R_\lambda R_\mu e_n = (\lambda_n - \lambda)^{-1}(\lambda_n - \mu)^{-1} e_n$$

where $(e_n)_{n\in\mathbb{N}}$ is a complete orthonormal system of eigenfunctions of Δ with $-\Delta e_n = \lambda_n e_n$. Then

$$(\lambda - \mu)\ \mathrm{tr}(R_\lambda R_\mu) = \sum_{n=0}^{\infty}\left(\frac{1}{s^2 - s_n^2} - \frac{1}{t^2 - s_n^2}\right).$$

Choosing $k(\delta) := \varphi_s(\delta) - \varphi_t(\delta)$ which is continuous for $\delta \geq 1$ and which satisfies $k(1) = -(s-t), \quad k(\delta) = O(\delta^{-2-\varepsilon})$ as $\delta \to \infty$ for some $\varepsilon > 0$. We have

$$g(x) = 2\pi \int_{\cosh(x)}^{\infty} k(u)\, du = \frac{1}{2} \int_{\cosh(x)}^{\infty} (\varphi_s - \varphi_t)(u)\, du$$

$$= \frac{1}{2} \int_{|x|}^{\infty} e^{-sy} - e^{-ty}\, dy = \frac{e^{-s|x|}}{2s} - \frac{e^{-t|x|}}{2t} \implies h(1+\ell^2) = \int_{\mathbb{R}} g(x) e^{itx}\, dx$$

$$= \frac{1}{s} \int_{\mathbb{R}} e^{i\ell x - s|x|}\, dx - \frac{1}{t} \int_{\mathbb{R}} e^{i\ell x - t|x|}\, dx = \frac{1}{s^2+\ell^2} - \frac{1}{t^2+\ell^2}$$

$$\implies h(\lambda_n) = \frac{1}{s^2 - s_n^2} - \frac{1}{t^2 - s_n^2} = \frac{1}{\lambda_n - \lambda} - \frac{1}{\lambda_n - \mu} \quad n \in \mathbb{N}.$$

We have that g is of rapid decay and since the series $\sum_{n=0}^{\infty} \lambda_n^{-2}$ is convergent we conclude that the series from (27) is convergent (by a quotient criteria). Applying the Selberg trace formula one can easily obtain the result. □

One can express the formula in a more geometric way. Using the properties stated in the first chapter we observe that

$$\sum_{m \in \mathbb{N}} h(\lambda_m) = \frac{\mathrm{vol}(\Gamma)}{4\pi^2} \int_{\mathbb{R}} h(1+t^2)\, t^2\, dt + \sum_{\{\gamma\}} \frac{g(\ell_\gamma)\ell_{\gamma_0}}{4|\mathcal{E}(\gamma)||\sinh(\ell_\gamma/2 + i\varphi/2)|^2}$$

$$= \frac{\mathrm{vol}(\Gamma)}{4\pi^2} \int_{\mathbb{R}} h(1+t^2)\, t^2\, dt + \sum_{\{\gamma\}:\ \ell_\gamma > 0\ \varphi \in 2\pi\mathbb{Z}} \frac{g(\ell_\gamma)\ell_{\gamma_0}}{4|\mathcal{E}(\gamma)||\sinh(\ell_\gamma/2)|^2}$$

$$+ \sum_{\{\gamma\}:\ \ell_\gamma > 0\ \varphi \notin 2\pi\mathbb{Z}} \frac{g(\ell_\gamma)\ell_{\gamma_0}}{4|\mathcal{E}(\gamma)||\sinh(\ell_\gamma/2 + i\varphi/2)|^2} + \sum_{\{\gamma\}:\ \ell_\gamma = 0\ \varphi \notin 2\pi\mathbb{Z}} \frac{g(0)\ell_{\gamma_0}}{4|\mathcal{E}(\gamma)||\sin(\varphi/2)|^2}.$$

Here in the first equality $\{\gamma\}$ runs over the Γ-conjugacy classes of hyperbolic or loxodromic elements and of elliptic elements, elements of Γ. In the second equality we have three terms, the first one is extended over the Γ-conjugacy classes of hyperbolic elements, the second one over the Γ-conjugacy classes of loxodromic elements and finally the third one over the Γ-conjugacy classes of ellitpic elements. Note that $\ell_{\gamma_0} = \log(N(T_0))$ is the minimal length of a hyperbolic or loxodromic element of $\mathcal{C}(\gamma)$. Define the primitive Γ-conjugacy classes of $\gamma \in \Gamma$ $\{\gamma\}_*$ to be the set of primitive elements of $\{\gamma\}$. An element is called primitive if it has minimal norm in the hyperbolic or loxodromic case and if it has minimal rotation angle in the elliptic case. Then the trace formula reads

$$\sum_{m \in \mathbb{N}} h(\lambda_m) = \frac{\mathrm{vol}(\Gamma)}{4\pi^2} \int_{\mathbb{R}} h(1+t^2)\, t^2\, dt + \sum_{\{\gamma\}_*\ \text{hyperbolic}} \sum_{n=1}^{\infty} \frac{g(n\ell_\gamma)\ell_\gamma}{4|\mathcal{E}(\gamma^n)||\sinh(n\ell_\gamma/2)|^2}$$

$$+\sum_{\{\gamma\}_* \text{ loxodromic}} \sum_{n=1}^{\infty} \sum_{k=1}^{m(\gamma)-1} \frac{g(n\ell_\gamma)\ell_\gamma}{4|\mathcal{E}(\gamma^n)||\sinh(n\ell_\gamma/2+ik\pi/m(\gamma))|^2}$$

$$+\sum_{\{\gamma\}_* \text{ elliptic}} \sum_{k=1}^{m(\gamma)-1} \frac{g(0)\ell_{\gamma_0}}{4|\mathcal{E}(\gamma^n)||\sin(k\pi/m(\gamma))|^2}.$$

Where we used $m(\gamma)=\mathrm{ord}(\gamma)$.

4.2 Huber's Theorem

Definition 4.1. Let $\Gamma < \mathbf{PSL}_2(\mathbb{C})$ be a discrete cocompact group (possibly containing elliptic elements). Suppose that $\ell_j = \log N(T_j)$ $(j \geq 0)$ are the logarithms of the norms of the hyperbolic or loxodromic elements of Γ arranged in stricly increasing order. Then the family of ordered pairs

$$\left(\ell_j, \sum_{\{T\}\text{lox.}} \sum_{\log N(T_j)=\ell_j} \frac{\ell_0}{4|\mathcal{E}(T_j)||\sinh(\log a(T_j))|^2}\right)_{(j\geq 1)}$$

is called the length spectrum of Γ.

Definition 4.2. If $\Gamma < \mathbf{PSL}_2(\mathbb{C})$ is a discrete cocompact group, the real number

$$E := \sum_{\{R\}\text{ ell.}} \frac{\log N(T_0)}{4|\mathcal{E}(R)||\sin(\frac{\pi k}{m(R)})|^2}$$

is called the elliptic number of Γ.

Theorem 4.3. *Let Γ_1, Γ_2 be discrete cocompact groups. Then the following hold:*

If the eigenvalue spectra for Γ_1 and Γ_2 coincide up to at most finitely many temrs then the eigenvalue spectra, length spectra, the volumes and the elliptic numbers for Γ_1 and Γ_2 coincide.

If the length spectra for Γ_1 and Γ_2 coincide up to at most finitely many temrs then the eigenvalue spectra, length spectra, the volumes and the elliptic numbers for Γ_1 and Γ_2 coincide.

Proof. Since the left-hand side of (27) for Γ_1 and Γ_2 agree up to finitely many terms (suppose that these terms are given by $s_{n_1}, ..., s_{n_k}$) we let s tend to infinity in the corresponding equation to see that the covolumes of Γ_1 and Γ_2 coincide. Let $T \to \infty$ in the equation (withou the volume terms now) which has the form

$$\sum_{k=1}^{m}(\pm)\frac{2s}{s^2-s_{n_k}^2} = E_1 - E_2 + \sum_{\{T\}_1 \text{ lox.}} c(T)N(T)^{-s} - \sum_{\{T\}_2 \text{ lox.}} c(T)N(T)^{-s}$$

where E_j is the elliptic number of Γ_j and $\{T\}_j$ means the summation over the conjugacy classes $\{T\}$ of hyperbolic or loxodromic elements of Γ_j. Letting s tend to infinity, we have $E_1 = E_2$. So the equation has the form

$$\sum_{k=1}^{m}(\pm)\frac{2s}{s^2 - s_{n_k}^2} = \sum_{\{T\}_1\ \{T\}_2 \text{lox.}} \gamma(T)N(T)^{-s}$$

where $\gamma(T)$ is the difference of the corresponding weights for Γ_1 and Γ_2. Assume that $\gamma(T) \neq 0$ and let $T^* \in \Gamma_1 \cup \Gamma_2$ be such that

$$N(T) := \min\{N(T)|\ \gamma(T) \neq 0\}.$$

Then we have

$$N(T^*)\sum_{k=1}^{m}(\pm)\frac{2s}{s^2 - s_{n_k}^2} = \gamma(T^*) + \sum_{N(T)>N(T^*)} \gamma(T)\left(\frac{N(T)}{N(T^*)}\right)^{-s}. \tag{27}$$

Fix σ so large that

$$\sum_{N(T)>N(T^*)} |\gamma(T)|\left(\frac{N(T)}{N(T^*)}\right)^{-\sigma} < \frac{|\gamma(T^*)|}{2}$$

and let $s = \sigma + i\tau$, $\tau \to \infty$. Then the left-hand side of (11) tends to zero whereas the right-hand side does not. Then $\gamma(T) = 0$ for all T and then the length spectra of Γ_1 and Γ_2 coincide. From this follows that the eigenvalue spectra for Γ_1 and Γ_2 coincide completely.
For the proof of (2) it yields similarly that the covolumes of Γ_1 and Γ_2 coincide so we omit these terms. We multiply the corresponding equation by $2s$ and compare the poles in the s-plane. The corresponding results yield similarly as in the first case. □

Two discrete cocompact gorups are said to be *isospectral* if their eigenvalue spactre (or length spectra) coincide. Theorem 4.3 has the following consequence.

Corollary 4.3.1. *Suppose that Γ_1, Γ_2 are discrete cocompact isospectral groups. Then either both Γ_1 and Γ_2 contain elliptic elements or neither of them contain elliptic elements.*

4.3 The Selberg Zeta Function

Let $\Gamma < \mathbf{PSL}_2(\mathbb{C})$ be a discrete cocompact group with fundamental domain $\mathcal{F}$. Suppose that

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq ...$$

are the eigenvalues of the operator $-\Delta : \mathbb{D} \to L^2(\Gamma \setminus \mathbb{H})$, and put

$$N := \max\{n \geq 0|\ \lambda_n < 1\}, \quad s_n := \sqrt{1-\lambda_n}\ n = 0, ..., Nn \quad s_n := i\sqrt{\lambda - 1}\ n \geq N+1.$$

Definition 4.3. For $\Re s > 1$, the Selberg zeta function for Γ is defined by

$$Z(s) := \prod_{\substack{\{T\} \in \mathcal{R} \\ k,h \geq 0 \\ k \equiv h \mod |\mathcal{E}(T)|}} (1 - a(T)^{-2k}\overline{a(T)^{-2h}}N(T)^{-s-1}) \tag{28}$$

where the product with respect to $\{T\}$ extends over a maximal reduce system $\mathcal{R}$ of Γ-conjugacy classes of the primitive hyperbolic or loxodromic elements of Γ. $\mathcal{R}$ is called reduced if no two of its elements have representatives with the same centralizer. The corresponding Selberg xi-function is

$$\Xi(s) := e^{\left(-\frac{v(\mathcal{F})}{6\pi}s^3 + Es\right)}Z(s),$$

where E is the elliptic number of Γ.

Using the geometric notation

$$Z(s) := \prod_{\substack{\{T\} \in \mathcal{R} \\ k,h \geq 0 \\ k \equiv h \mod |\mathcal{E}(T)|}} (1 - e^{-\ell_T(k+h+s+1)+i\varphi_T(h-k)}) = \prod_{\substack{\{T\} \in \mathcal{R} \\ k,h \geq 0 \\ k \equiv h \mod |\mathcal{E}(T)|}} (1 - e^{-\ell_T(k+h+s+1)})$$

where ℓ_T is the displacement length of T and φ_T is the rotation angle of T. For $\sigma = \Re s > 1$

$$\sum_{\substack{\{T\} \in \mathcal{R} \\ k,h \geq 0}} |e^{-\ell_T(k+h+s+1)+i\varphi_T(h-k)}| = \sum_{\substack{\{T\} \in \mathcal{R} \\ k,h \geq 0}} |e^{-\ell_T(k+h+\sigma+1)}|$$

$$= \sum_{\substack{\{T\} \in \mathcal{R} \\ m \geq 0}} (m+1)e^{-\ell_T(m+1+\sigma)} = \sum_{\{T\} \in \mathcal{R}} \frac{e^{-\ell_T(1+\sigma)}}{(1-e^{-\ell_T})^2} = \sum_{\{T\} \in \mathcal{R}} \frac{N(T)^{-1-\sigma}}{(1-|a(T)|^{-2})^2}$$

$$\leq \sum_{\{T\} \text{ lox.}} N(T)^{-1-\sigma} = \int_1^\infty x^{-1-\sigma}\pi_0(x)\, dx \leq C\int_1^\infty x^{1-\sigma}\, dx = C\frac{1}{\sigma-1}$$

for C given in section 2.6. Then we conclude that Z and Ξ are nowhere vanishing, holomorphic functions defined in the half-plane $\Re s > 1$. By Corollary 4.2.1 we have

$$\sum_{\{T\} \text{ lox.}} \frac{\log N(T_0)}{4|\mathcal{E}(T)||\sinh(\log a(T))|^2}N(T)^{-s} \sim \frac{1}{s-1} \quad \text{for } s \to 1^+.$$

By Karamata's tauberian theorem we obtain

$$\sum_{\substack{\{T\} \text{ lox.} \\ N(T) \leq x}} \frac{\log N(T_0)}{4|\mathcal{E}(T)||\sinh(\log a(T))|^2}N(T)^{-1} \sim \log x \quad \text{for } x \to \infty.$$

Which leads us to conclude

$$\sum_{\substack{\{T\}\text{ lox.}\\ N(T)\leq x}} \frac{\log N(T_0)}{4|\mathcal{E}(T)|N(T)^2} \sim \log x \quad \text{for } x\to\infty$$

since $|a(T)| \to \infty$ as $\{T\}$ runs through the conjugacy classes of the hyperbolic or loxodromic elements of Γ.

Lemma 4.4. *For $\Re s > 1$, we have*

$$\frac{d}{ds}\log Z(s) = \frac{Z'(s)}{Z(s)} = \sum_{\{T\}\text{ lox.}} \frac{\log N(T_0)}{|\mathcal{E}(T)||a(T)-a(T)^{-1}|^2} N(T)^{-s}.$$

Proof.

$$\sum_{\{T\}\text{ lox.}} \frac{\log N(T_0)}{|\mathcal{E}(T)||a(T)-a(T)^{-1}|^2} N(T)^{-s}$$

$$= \sum_{\substack{\{T\}\in\mathcal{R}\\ n\geq 0\\ 1\leq\nu\leq|\mathcal{E}(T)|}} \frac{\log N(T_0)}{|\mathcal{E}(T_0)||e^{\varphi_{T_0}i\nu/2}a(T_0)^{n+1}-e^{-\varphi_{T_0}i\nu/2}a(T_0)^{-n-1}|^2} N(T_0)^{-s(n+1)}$$

$$= \sum_{\substack{\{T\}\in\mathcal{R}\\ n\geq 0\\ 1\leq\nu\leq|\mathcal{E}(T)|}} \frac{N(T_0)^{-(s+1)(n+1)}\log N(T_0)}{|\mathcal{E}(T_0)|\left(1-e^{-\nu\varphi_{T_0}i}a(T_0)^{-2(n+1)}\right)\left(1-e^{\nu\varphi_{T_0}i}\overline{a(T_0)}^{-2(n+1)}\right)}$$

$$= \sum_{\substack{\{T\}\in\mathcal{R}\\ k,h,n\geq 0\\ k\equiv h \mod |\mathcal{E}(T_0)|}} a(T_0)^{-2k(n+1)}\overline{a(T_0)}^{-2h(n+1)} N(T_0)^{-(s+1)(n+1)}\log N(T_0)$$

$$\sum_{\substack{\{T\}\in\mathcal{R}\\ k,h\geq 0\\ k\equiv h \mod |\mathcal{E}(T_0)|}} \frac{a(T_0)^{-2k}\overline{a(T_0)}^{-2h}N(T_0)^{-(s+1)}\log N(T_0)}{1-a(T_0)^{-2k}\overline{a(T_0)}^{-2h}N(T_0)^{-(s+1)}} = \frac{Z'(s)}{Z(s)}.$$

□

Theorem 4.5. *Let the hypotheses and notations be as in Definition 4.3. Then the Selberg zeta function and the xi-function, defined for $\Re s > 1$ are entire functions of s, and*

$$\frac{1}{2s}\frac{\Xi'(s)}{\Xi(s)} - \frac{1}{2t}\frac{\Xi'(t)}{\Xi(t)} = \sum_{n=0}^{\infty}\left(\frac{1}{s^2-s_n^2} - \frac{1}{t^2-s_n^2}\right) \tag{29}$$

for all $s,t\in\mathbb{C}\setminus\{\pm s_n:\ n\geq 0\}$. The zeros of Z and Ξ are the numbers $\pm s_n$, $n\geq 0$. If $\lambda_n \neq 1$, both the numbers s_n, $-s_n$ are zeros whose multiplicities are equal to the multiplicity of the eigenvalue λ_n. If $\lambda_k = 1$ is an eigenvalue of $-\Delta : \mathcal{D}\to L^2(\Gamma\setminus\mathbb{H})$, then $s_k = 0$ is a zero whose multiplicity equals twice the multiplicity of the eigenvalue $\lambda_k = 1$.

Proof. From the stated results on the Selberg zeta funciont we deduce that (30) holds for $\Re s > 1,\ \Re t > 1$. Now let t be fixed, $\Re t > 1$. Then the right-hand side of (30) is a meromorphic function of $s \in \mathbb{C}$ with simple poles only at $\pm s_n$. Hence the $\frac{\Xi'}{\Xi}$ is a meromorphic function with simple poles only at the points $\pm s_n$. Since

$$\frac{2s}{s^2 - s_n^2} = \frac{1}{s + s_n} + \frac{1}{s - s_n},$$

we conclude: For $\lambda_n \neq 1$, the residue of $\frac{\Xi'}{\Xi}$ at $\pm s_n$ is equal to the multiplicity of λ_n, and if $\lambda_k = 1$ is an eigenvalue, the residue of the pole $s_k = 0$ equals twice the multiplicity of the eigenvalue $\lambda_k = 1$. Since all the residues are non-negative integers, we see that Ξ is an entire function whose zeros are as described in the Theorem. □

Corollary 4.5.1. *The Selberg zeta function and the xi-function satisfy the functional equations*

$$Z(-s) = e^{\left(-\frac{v(\mathcal{F})}{3\pi}s^3 + 2Es\right)}\ Z(s),$$
$$\Xi(-s) = \Xi(s).$$

Proof. By the last theorem we have that $\frac{1}{2s}\frac{\Xi'(s)}{\Xi(s)}$ is invariant with respect to $s \to -s$. Since $\frac{\Xi'}{\Xi}$ is an odd function, it follows that the logarithmic derivative of $\frac{\Xi(s)}{\Xi(-s)}$ equals zero. Hence $\frac{\Xi(s)}{\Xi(-s)}$ is constant. From the last theorem we have that the order of Ξ at zero is even. Hence

$$\lim_{s \to 0} \frac{\Xi(s)}{\Xi(-s)} = 1.$$

□

4.4 Weyl's Asymptotic Law

Suppose $\Re s > 1$ is real. Then we have $Z(s) > 0$ for $s > 1$ and in particular there are no zeros in the region

$$\{s \in \mathbb{C} :\ \Re s > 0,\ s \notin]0, 1]\}$$

by Theorem 4.5. Then there exists a holomorphic logarithm $\log Z$ of Z in this region which is uniquely determined if we require

$$\log Z(s) \in \mathbb{R},\quad s \in \mathbb{R},\ s > 1.$$

We denote by $\log Z$ the holomorphic logarithm. Since Z does not has a zero in the region $\mathbb{C} \setminus O := \{it : t_n < |t| < t_{n+1},\ n \geq N + 1,\ t \in \mathbb{R} \setminus \{0\}\}$ $\log Z$ has a unique continuous extension to this region (which we also denote by $\log Z$). For $s \in \mathbb{C} \setminus (O \cup]0, 1])$, $\Re s > 1$ we put

$$\arg Z(s) := \Im(\log Z(s)).$$

Since we are concerned with the asymptotic behaviour of the number of eigenvalues less than a fixed quantity we define

$$\mathcal{N}(\lambda) := |\{\lambda_n \leq \lambda : \; n \geq 0\}|$$

$$A(T) := |\{t_n \leq T : \; n \geq N+1\}|, \quad A(\sqrt{T}) + N = \mathcal{N}(T)$$

for $\lambda \geq 0, \; T > 0$.

Theorem 4.6. *Suppose that $T > 0$, $T \neq t_n$ for all $n \geq N+1$. Then*

$$A(T) = \frac{v(\mathcal{F})}{6\pi^2} T^3 + \frac{E}{\pi} T + \frac{1}{\pi} \arg Z(iT) - N. \tag{30}$$

Proof. By Theorem 4.5 we have

$$2(A(T) + N) = \frac{1}{2\pi i} \int_{\partial R(T)} \frac{Z'}{Z}(s) \; ds,$$

where $R(T)$ is the rectangle with vertices $2 + iT, \; -2 + iT, \; -2 - iT, \; 2 - iT$. The boundary of $R(T)$ splits into two parts $R^+(T), \; R^-(T)$, situated in the half-planes $\Re s \geq 0$ and $\Re s \leq 0$, respectively. The transformation $s \to -s$ maps $R^-(T)$ onto $R^+(T)$ such that the orientation is preserved. Hence we deduce from the functional equation

$$\begin{aligned}
\frac{1}{2\pi i} \int_{\partial R(T)} \frac{Z'}{Z}(s) \; ds &= \frac{1}{2\pi i} \int_{R^+(T)} \frac{Z'}{Z}(s) \; ds - \frac{1}{2\pi i} \int_{R^+(T)} \frac{Z'}{Z}(-s) \; ds \\
&= \frac{1}{2\pi i} \int_{R^+(T)} \left(-\frac{v(\mathcal{F})}{\pi} s^2 + 2E \right) \; ds + \frac{1}{\pi i} \int_{R^+(T)} \frac{Z'}{Z}(-s) \; ds \\
&= \frac{v(\mathcal{F})}{6\pi^2} T^3 + \frac{E}{\pi} T + \frac{1}{\pi} \arg Z(iT).
\end{aligned}$$

□

Lemma 4.7. *Suppose that $\sigma < 0$. Then there exists a bounded function $f_\sigma : \mathbb{R} \to \mathbb{C}$ such that*

$$Z(\sigma + it) = f_\sigma e^{\left(\frac{v(\mathcal{F})}{\pi} |\sigma| t^2\right)} \; Z(-\sigma - it), \quad \forall t \in \mathbb{R}.$$

Proof. It follows from the functional equation. □

Lemma 4.8. *Z is an entire function of order 4.*

Proof. Since $\sum_{n \in \mathbb{N}} \lambda_n^{-2} < \infty$ we have

$$\sum_{n=0}^{\infty}{}' |s_n|^{-4} < \infty,$$

where the primed sum indicates omission of possible terms $s_n = 0$. Let $p \in \{0,1,2,3\}$ be the minimal integer such that

$$\sum_{n=0}^{\infty}{}' |s_n|^{-p-1} < \infty,$$

and let $k \geq 0$ denote the multiplicity of the eigenvalue 1 of $-\Delta : \mathcal{D} \to L^2(\Gamma \setminus \mathbb{H})$. The canonical product

$$\Phi(s) := s^{2k} \prod_{n=0}^{\infty}{}' \left(1 - \frac{s}{s_n}\right) e^{\left(\frac{s}{s_n} + \ldots + \frac{1}{p}\left(\frac{s}{s_n}\right)^p\right)} \cdot \prod_{n=0}^{\infty}{}' \left(1 + \frac{s}{s_n}\right) e^{\left(-\frac{s}{s_n} + \ldots + \frac{1}{p}\left(-\frac{s}{s_n}\right)^p\right)}$$

in an entire function of order equal to the exponent of the convergent series

$$\sum_{n=0}^{\infty}{}' |s_n|^{-\alpha}$$

which is at most equal to 4 ([17]). Φ has the same zeros as Z, and

$$\frac{Z'}{Z} - \frac{\Phi'}{\Phi}$$

is a polynomial of degree at most 2. This implies

$$Z(s) = \Phi e^{q(s)},$$

where q is a polynomial of degree at most 3. Hence Z is an entire function of order at most 4. □

Corollary 4.8.1. *Suppose that $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$. Then there exists a constants $C > 0$ such that*

$$Z(\sigma + it) = O(e^{Ct^2}) \quad as \quad |t| \to \infty \tag{31}$$

uniformly with respect to $\sigma \in [\alpha, \beta]$.

Proof. We can assume that $\alpha < -1 < 1 < \beta$. Then $Z(\beta + it)$ is a bounded function of $t \in \mathbb{R}$. By Lemma 4.7 this implies that (32) holds for $\sigma = \alpha$ and for $\sigma = \beta$. Since Z is of finite order by Lemma 4.8 the Phragmén-Lindelöf theorem (citecomplex an) yields the assertion. □

Theorem 4.9. *Suppose that $T > 0$, $T \neq t_n$ for all $n \geq N+1$. Then*

$$\arg Z(iT) = O(T^2) \quad as \ T \to \infty.$$

Proof. Let $P(T)$ be the polygonal curve consisting on the line segments $Q(T)$ from 2 to $2+iT$ and $R(T)$ from $2+iT$ to iT. Then

$$\arg Z(iT) = \Im \int_{P(T)} \frac{Z'}{Z}(s)\ ds = \arg Z(2+iT) + \Im \int_{R(T)} \frac{Z'}{Z}(s)\ ds.$$

For $\Re s > 1$

$$\begin{aligned}
-\log Z(s) &= \sum_{\substack{\{T\} \in \mathcal{R} \\ k,h \geq 0,\ n \geq 1 \\ k \equiv h \mod |\mathcal{E}(T)|}} \frac{1}{n}\ a(T)^{-2nk}\overline{a(T)}^{-2nh} N(T)^{-n(s+1)} \\
&= \sum_{\substack{\{T\} \in \mathcal{R} \\ k,h \geq 0,\ n \geq 1 \\ 1 \leq \nu \leq |\mathcal{E}(T)|}} \frac{e^{-\varphi_T \nu(k-h)} a(T)^{-2nk}\overline{a(T)}^{-2nh}}{n|\mathcal{E}(T)|} N(T)^{-n(s+1)} \\
&= \sum_{\substack{\{T\} \in \mathcal{R} \\ k,h \geq 0,\ n \geq 1 \\ 1 \leq \nu \leq |\mathcal{E}(T)|}} \frac{1}{n} \frac{1}{|\mathcal{E}(T)|} \frac{1}{|1 - a(T)^{-2n} e^{-\varphi_T \nu}|^2} - N(T)^{-n(s+1)}
\end{aligned}$$

the right-hand side converges absolutely and attains real values for real $s > 1$. Choosing $0 < \alpha < 1$ such that $|a(T)|^{-2} \leq \alpha$ for all primitive hyperbolic or loxodromic elements of Γ we have that for $\sigma = \Re s > 1$

$$|\log Z(s)| \leq (1-\alpha)^{-2} \sum_{\substack{\{T\} \in \mathcal{R} \\ n \geq 1}} \frac{1}{n} N(T)^{-n(\sigma+1)} = (1-\alpha)^{-2} \sum_{\{T\} \in \mathcal{R}} \log \frac{1}{1 - N(T)^{-\sigma-1}}.$$

The right-hand side is finite. Hence

$$\arg Z(2+iT) = O(1) \quad \text{as } T \to \infty.$$

For the second term we have that

$$\Im \int_{R(T)} \frac{Z'}{Z}(s)\ ds$$

is the increment of the argument of $Z(S)$ as s runs through the segment from $2+iT$ to iT. Each time the argument of $Z(s)$ changes by a quantity of absolute value at least π, $\Re Z(s)$ undergoes a change of sign. If $c(T)$ is the number of times $\Re Z(\sigma + iT)$ changes sign as σ decreases from 2 to zero

$$\left| \Im \int_{R(T)} \frac{Z'}{Z}(s)\ ds \right| \leq \pi(c(T)+2).$$

Setting $\theta_T(w) := \Re Z(w+iT)$ we see that $c(T)$ equals the number $\nu(T)$ of zeroes of the entire function $\theta_T(w)$ in $[0,2]$ up to an error term no exceeding 2 due to the possible zeros at 0 and 2. By Jensen's formula ([17]) (applied ti the disc of center 0 and radius 3) we have

$$\nu(T)\log\frac{3}{2} \leq \int_0^{2\pi} \log|\theta_T(3e^{it})|\; dt - \log|\theta_T(0)|.$$

If 0 is a zero of $\theta_T(w)$ a slight move of the center of our disc yields the same conclusion. By Corollary 4.8.1 we have $\nu(T) = O(T^2)$ as $T \to \infty$. □

The next result follows directly from Theorem 4.9.

Theorem 4.10. *Suppose that $T > 0$, $T \neq t_n$ for all $n \geq N+1$. Then the counting function $A(T)$ satisfies*

$$A(T) = \frac{v(\mathcal{F})}{6\pi^2}T^3 + O(T^2), \qquad as\ T \to \infty.$$

□

Corollary 4.10.1. *The series*

$$\sum_{n=0}^{\infty}{}' |s_n|^{-\alpha} \qquad \alpha \in \mathbb{R},$$

converges if and only if $\alpha > 3$. We have

$$\sum_{0<t_n\leq T}{}' |s_n|^{-3} = \frac{v(\mathcal{F})}{2\pi^2}\ \log T + O(1) \qquad as\ T \to \infty.$$

Proof. By partial summation: For $n > m \geq N+1,\ t_m \neq 0,\ t_n < T < t_{n+1}$, we obtain

$$\sum_{n=0}^{\infty} |s_n|^{-\alpha} = [x^{-\alpha}A(x)]_{t_m}^T + \alpha \int_{t_m}^T x^{-\alpha-1}A(X)\; dx.$$

The result follows from Theorem 4.10. □

Theorem 4.11. *The functions Z and Ξ are entire functions of order 3.*

□

The last theorem implies

$$\Phi(s) = s^{2k} \prod_{n=0}^{\infty}{}' \left(1 - \left(\frac{s}{s_n}\right)^2\right)\ e^{(s/s_n)^2}$$

$$\implies \frac{1}{2s}\frac{\Phi'}{\Phi}(s) = \frac{k}{s^2} + \sum_{n=0}^{\infty}{}' \left(\frac{1}{s^2 - s_n^2} + \frac{1}{s_n^2}\right).$$

Where $k \geq 0$ is the multiplicity of the eigenvalue 1 of $\Delta : \mathcal{D} \to L^2(\Gamma \setminus \mathbb{H})$, and the prime indicates that factors with $s_n = 0$ must be omitted. Then, for $t \neq s_n,\ n \geq 0$ it follows that

$$\frac{1}{2s}\frac{\Xi'}{\Xi}(s) - \frac{1}{2s}\frac{\Phi'}{\Phi}(s) = \sum_{n=0}^{\infty}\left(\frac{1}{s^2 - s_n^2} - \frac{1}{t^2 - s_n^2}\right) + \frac{1}{2s}\frac{\Xi'}{\Xi}(t)$$

$$-\frac{1}{2s}\left(\frac{2k}{s} + \sum_{n=0}^{\infty}{}'\left(\frac{2s}{s^2 - s_n^2} + \frac{2s}{s_n^2}\right)\right) = \frac{1}{2s}\frac{\Xi'}{\Xi}(t) - \frac{k}{t^2}\sum_{n=0}^{\infty}{}'\left(-\frac{1}{t^2 - s_n^2} + \frac{1}{s_n^2}\right)$$
$$= \frac{1}{2t}\frac{\Xi'}{\Xi}(t) - \frac{1}{2t}\frac{\Phi'}{\Phi}(t).$$

Corollary 4.11.1. *There exist real constants α, β such that Ξ and Z have canonical factorisations of the form*

$$\Xi(s) = \alpha\ e^{\beta s^2}\ \Phi(s),$$
$$Z(s) = \alpha e^{\left(\frac{v(\mathcal{F})}{6\pi}s^3 + \beta s^2 - Es\right)}\ \Phi(s).$$

Looking at the Taylor series of Z, Ξ, Φ we obtain

$$\alpha = \lim_{s\to 0} s^{-2k}\Xi(s) = \lim_{s\to 0} s^{-2k}Z(s),$$

$$\beta = \lim_{s\to 0}\frac{\alpha^{-1}s^{-2k}\Xi(s) - 1}{s^2} = \lim_{s\to 0}\frac{1}{2s}\left(\frac{\Xi'}{\Xi}(s) - \frac{2k}{s}\right).$$

There's not a single example of a discrete cocompact group Γ known where the sequence $(\lambda_n)_{n\geq 0}$ is known explicitly.

4.5 The Prime Geodesic Theorem

Primitive elements of Γ play a similar rôle to that of the prime numbers. We proceed in a similar way than the Prime Number Theorem defining

$$\pi_{00}(x) := |\{\{T_0\}_\Gamma :\ T_0 \text{ primitive hyperbolic or loxodromic},\ N(T_0) \leq x\}|$$

and we call the prime geodesic theorem the result which describes the asymptotic behaviour of π_{00}. Let

$$\Lambda(T) := \frac{\log N(T_0)}{|\mathcal{E}(T)||a(T) - a(T)^{-1}|^2}$$

for T a hyperbolic or loxodromic element of Γ. Then we have

$$\frac{d}{ds}\log Z(s) = \sum_{\{T\}\ \text{lox.}} \Lambda(T)\ N(T)^{-s}.$$

Define

$$\Psi(x) := \sum_{\substack{\{T\} \text{ lox.} \\ N(T) \leq x}} \Lambda(T), \quad \Theta(x) := \sum_{\substack{\{T_0\} \text{ lox.} \\ N(T_0) \leq x}} \frac{\log N(T_0)}{N(T_0)}$$

where the summation of the function Θ extends over all Γ-conjugacy classes $\{T_0\}$ of primitive hyperbolic or loxodromic elements of Γ with norm $N(T_0) \leq x$.

Lemma 4.12.

$$\Theta(x) - \Psi(x) = O(\log x) \quad \text{for } x \to \infty.$$

Proof. Taking $C := \max\limits_{\{T\} \text{ lox}} \{|1 - a(T)^{-2}|^{-1}\}$ we have

$$|\Psi(x) - \sum_{\substack{\{T_0\} \text{ lox.} \\ N(T_0) \leq x}} \Lambda(T_0)| \leq C \left(\sum_{\substack{\{T_0\} \text{ lox.} \\ N(T_0) \leq x}} \frac{\log N(T_0)}{N(T_0)^2} + \sum_{\substack{\{T_0\} \text{ lox.} \\ N(T_0) \leq x}} \frac{\log N(T_0)}{N(T_0)^3} + ... \right).$$

Then

$$\sum_{\substack{\{T_0\} \text{ lox.} \\ N(T_0) \leq x}} \frac{\log N(T_0)}{N(T_0)^2} = O(\log x) \ \text{ for } x \to \infty.$$

For all $k \geq 3$

$$\sum_{\substack{\{T_0\} \text{ lox.} \\ N(T_0) \leq x}} \frac{\log N(T_0)}{N(T_0)^k} \leq \sum_{\substack{\{T_0\} \text{ lox.} \\ N(T_0) \leq x}} \frac{\log N(T_0)}{N(T_0)^3} < \infty.$$

Hence

$$\Phi(x) - \sum_{\substack{\{T_0\} \text{ lox.} \\ N(T_0) \leq x}} \Lambda(T_0) = O(\log x) \quad \text{for } x \to \infty.$$

Since there exist only finitely many primitive Γ-conjugacy classes $\{T_0\}$ such that $|\mathcal{E}(T_0)| \neq 1$ and

$$||a(T) - a(T)^{-1}|^{-2} - N(T_0)^{-1}| = N(T_0)^{-1}||1 - a(T)^{-2}|^{-2} - 1| \leq (1 + C^2)N(T_0)^{-2}$$

for all primitive hyperbolic or loxodromic elements of Γ we obtain

$$\sum_{\substack{\{T_0\} \text{ lox.} \\ N(T_0) \leq x}} \frac{\log N(T_0)}{|a(T_0) - a(T_0)^{-1}|^2} - \Theta(x) = O(\log x) \quad \text{for } x \to \infty.$$

We finally have

$$\left| \sum_{\substack{\{T_0\} \text{ lox.} \\ N(T_0) \leq x}} \frac{\log N(T_0)}{|a(T_0) - a(T_0)^{-1}|^2} - \Theta(x) \right| \leq (1 + C^2) \sum_{\substack{\{T_0\} \text{ lox.} \\ N(T_0) \leq x}} \frac{\log N(T_0)}{N(T_0)^2} = O(\log x)$$

for $x \to +\infty$. □

Theorem 4.13. *For $x \to \infty$ we have*

$$\Psi(x) \sim x, \quad \Theta \sim x.$$

Proof. Since

$$\frac{d}{ds} Z(s) - \frac{1}{s-1}$$

converges to a finite limit for $\Re s \to 1$. This convergence is uniform in every interval $|\Im s| \leq \alpha$. The asymptotic relation holds by Ikehara's theorem. □

Theorem 4.14. (Prime Geodesic Theorem)

$$\pi_{00}(x) \sim \frac{x^2}{\log(x^2)} \quad \text{for } x \to \infty.$$

Proof. For $\alpha > 1$ so small that $\Theta(t) = 0$ for $1 \leq 1 \leq \alpha$. we have

$$\pi_{00}(x) = \int_\alpha^x \frac{t}{\log t}\, d\Theta(t) = \left[\frac{t\Theta(t)}{\log t}\right]_\alpha^x - \int_\alpha^x \frac{\Theta(t)}{\log t}\, dt + \int_\alpha^x \frac{\Theta(t)}{(\log t)^2}\, dt.$$

By the last theorem

$$\int_\alpha^x \frac{\Theta(t)}{\log t}\, dt \sim \int_\alpha^x \frac{t}{\log t}\, dt = \int_{\alpha^2}^{x^2} \frac{du}{\log u} \sim \ell i(x^2) \sim \frac{x^2}{\log x^2}$$

for $x \to \infty$. We defined the integral logarithm as the principal value

$$\ell i(x) := \lim_{\delta \to 0} \left(\int_0^{1-\delta} \frac{dt}{\log t} + \int_{1+\delta}^1 \frac{dt}{\log t} \right).$$

We use the well-known fact that

$$\ell i(x) \sim \frac{x}{\log x} \quad \text{for } x \to \infty.$$

Related on the second integtral on the right-hand side of $\pi_{00}(x)$

$$\int_\alpha^x \frac{\Theta(t)}{(\log t)^2}\, dt \sim \int_\alpha^x \frac{t}{(\log t)^2} = 2\int_\alpha^{x^2} \frac{du}{(\log u)^2} = 2\int_\alpha^x \frac{du}{(\log u)^2} + 2\int_x^{x^2} \frac{du}{(\log u)^2}$$

$$= O(\ell i(x)) + O\left(\frac{1}{\log x}\int_x^{x^2} \frac{du}{\log u}\right) = O(\ell i(x)) + O\left(\frac{\ell i(x^2)}{\log x}\right) = O\left(\frac{x^2}{(\log x)^2}\right).$$

□

Lemma 4.15. *The counting function π_0 of Lemma 2.8 satisfies*

$$\pi_0(x) = \pi_{00}(x) + O(\frac{x}{\log x}) \quad \text{for } x \to \infty.$$

Proof.

$$\pi_0(x) = \pi_{00}(x) + \pi_{00}\left(x^{1/2}\right) + ... + \pi_{00}\left(x^{1/n(x)}\right)$$

where $n(x) + 1$ is the smallest integer such that $\pi_{00}\left(x^{1/(n(x)+1)}\right) = 0$. Then

$$n(x) = O(\log x), \qquad \pi_{00}(x^{1/2}) = O\left(\frac{x}{\log x}\right),$$

and for $3 \leq k \leq n(x)$

$$\pi_{00}\left(x^{1/k}\right) \leq \pi_{00}\left(x^{1/3}\right) = O\left(\frac{x^{2/3}}{\log x}\right).$$

Hence

$$\pi_0(x) = \pi_{00}(x) + O\left(\frac{x}{\log x}\right) \qquad \text{for } x \to \infty.$$

□

Corollary 4.15.1. *The counting function π_0 satisfies*

$$\pi_0(x) \sim \frac{x^2}{\log(x^2)} \qquad \text{for } x \to \infty.$$

5 Miscellanea

In this chapter we define some important symmetric functions used in spectral geometry and we give a few properties of the Riemann zeta function. Some properties of the Selberg zeta function are similar to the Riemann zeta function ones. We follow [18], [29], and [5].

5.1 Spectral Geometry

Let (M, g) be a compact Riemannian manifold of dimension m given by a compact topological space M with a differentiable structure and g the Riemannian metric. We use the same notation as in the introduction of the first chapter

The measure dv given by the metric g induces the following scalar product:

$$\langle f, g \rangle := \int_M f\overline{g}\ dv \quad \forall f, g \in L^2(M)$$

which makes $L^2(M)$ a Hilbert space. Let $H_0^2(M) := \{\ f \in L^1_{loc}(M)|\ f = 0\ in\ \partial M \subset \mathbb{R}^{2m+1},\ f \in L^2(M),\ \exists \partial^\alpha f \in L^2(M)\ \forall |\alpha| \leq 2\}$.

Theorem 5.1. *The Laplace operator is self-adjoint and there exists a Hilbert basis $\{\varphi_n\}_{n\in\mathbb{N}}$ of $L^2(M)$ and a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ with $\lambda_n \to \infty$ as $n \to \infty$ such that:*

$$\Delta\varphi_n = \lambda_n\varphi_n\ in\ M$$

$$\varphi_n \in H_0^2(M) \cap C^\infty(M)\ \forall n \in \mathbb{N}.$$

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \to \infty\ is\ the\ spectrum\ of\ \Delta$$

Definition 5.1. We define the following semigroup

$$\mathrm{e}^{-t\Delta} := \sum_{n=0}^{\infty} \frac{t^n(-\Delta)^n}{n!} \quad \forall t \geq 0$$

Note that by the definition of $\mathrm{e}^{-t\Delta}$ we obtain

$$\mathrm{e}^{-t\Delta}\varphi_k = \sum_{n=0}^{\infty} \frac{t^n(-\Delta)^n\varphi_k}{n!} = \sum_{n=0}^{\infty} \frac{t^n(-\lambda_k)^n}{n!}\varphi_k = e^{-\lambda_k t}\varphi_k \quad \forall k \in \mathbb{N} \quad \forall t \geq 0.$$

Lemma 5.2. $\left(\mathrm{e}^{-t\Delta}\right)_{t\geq 0}$ *is a strongly continuous sempigroup.*

Definition 5.2. The heat kernel

$$K(t, x, y) : (0, +\infty) \times M \times M \longrightarrow \mathbb{C}$$

is the fundamental solution of the heat operator, i.e.

$$\begin{cases} \frac{\partial}{\partial t}K(t,x,y) = -\Delta_x K(t,x,y) \;\; \forall\, t \geq 0 \;\; \forall x,y \in M \\ \\ \lim_{t\to 0^+} K(t,x,y) = \delta(x-y) \;\; \forall x,y \in M \end{cases}$$

$K \in \mathbb{C}^\infty(\mathbb{R}_+ \times M \times M)$. Δ_x is the Laplace operator acting on the variable x.

Proposition 5.1. *Assume that there exists the heat kernel of* (M,g), $K(t,x,y)$. *Then we have the pointwise convergence*

$$K(t,x,y) = \sum_{n=0}^{\infty} e^{-t\lambda_n}\varphi_n(x)\overline{\varphi}_n(y).$$

Proposition 5.2. (Minakshisundaram-Pleijel expansion [7]) *There exist* $u_k \in \mathcal{C}^\infty(U_r),\; k \in \mathbb{N},\; U_r := \{(x,y) \in M \times M :\; d(x,y) < r\}$ *such that heat kernel has the asymptotic expansion*

$$K(t,x,y) \sim \frac{1}{(4\pi t)^{m/2}} \sum_{k=0}^{\infty} u_k(x,x)t^k.$$

The distance between $x,y \in M$ $d(x,y)$ is defined as the length of the geodesic which goes from x to y.

Definition 5.3. (Spectral Counting function)

$$\mathcal{N}(\lambda) := \sum_{\lambda_n \leq \lambda} 1 \quad \lambda > 0.$$

Definition 5.4. (Spectral Theta function)

$$\theta_M(t) := \operatorname{tr}\left(\mathrm{e}^{-t\Delta}\right) = \sum_{n=0}^{\infty} e^{-t\lambda_n} = \int_0^\infty e^{-\lambda}\, d\mathcal{N}(\lambda) \;\; \forall t > 0.$$

Note that by the spectral mapping theorem:

$$\sigma(\mathrm{e}^{-t\Delta}) = \mathrm{e}^{-t\sigma(\Delta)}.$$

Theorem 5.3.

$$\theta_M(t) \sim \frac{1}{(4\pi t)^{m/2}} \sum_{k=0}^{\infty} a_k t^k,$$

where

$$a_k = \int_M u_k(x,x)dv(x).$$

Definition 5.5. (Spectral Zeta function)

$$\zeta_M(s) := \mathrm{tr}\left(\Delta^{-s}\right) = \sum_{n=1}^{\infty} \lambda_n{}^{-s} = \int_0^{\infty} \lambda^{-s}\, d\mathcal{N}(\lambda)$$

where the sum is taken over the non-zero eigenvalues.

Theorem 5.4. *$\zeta_M(s)$ converges and is holomorphic for $\Re s > \frac{m}{2}$, and has a meromorphic continuation to $\mathbb{C}$ with at worst simple poles, ocurring only at $s = \frac{m}{2}, \frac{m}{2}-1, \frac{m}{2}-2, ..., \frac{m}{2} - \lceil\frac{m-1}{2}\rceil$. In particular,*

$$\zeta_M(0) = \begin{cases} -\dim\ker\Delta, & m \text{ odd}, \\ \int_M u_{m/2} - \dim\ker\Delta, & m \text{ even}. \end{cases}$$

Then by Ikehara's tauberian theorem we have:

Theorem 5.5. (Weyl's law) *Let M be a compact Riemannian manifold of dimension m. Then the eigenvalues of the Laplace operator satisfy the following asymptotic formula:*

$$\lambda_n \sim n^{2/m} \frac{4\pi^2}{\left(\mathrm{vol}(M)\Gamma(\frac{m}{2}+1)\right)^{m/2}} \quad n \to \infty.$$

In the case of a finite-dimensional Hilbert space (for example $\mathbb{C}^n$) the determinant of a diagonalizable operator A (a diagonalizable matrix of egienvalues λ_k) is:

$$\det A = \prod_{k=1}^{n} \lambda_k.$$

To extend this property to the infinite-dimensional case we observe that:

$$\zeta_M'(s) = -\sum_{n=1}^{\infty} \log\lambda_n\ \lambda_n^{-s} \Rightarrow e^{-\zeta_M'(s)} = \prod_{n=1}^{\infty} \lambda_n^{\lambda_n^{-s}} \quad \forall s \in \mathbb{C}.$$

Definition 5.6. (Spectral Determinant) We define the zeta-regularized determinant of Δ

$$\det{}_\zeta(\Delta) := e^{-\zeta_M'(0)} = \prod_{n=1}^{\infty} \lambda_n.$$

Let

$$E_{m,M}(\lambda,\mu) := \left(1 - \frac{\lambda}{\mu}\right)\ e^{\left(\frac{\lambda}{\mu} + \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2 + \ldots + \frac{1}{m-1}\left(\frac{\lambda}{\mu}\right)^{m-1}\right)},$$

where $m > 1$ is an integer, $\lambda, \mu \in \mathbb{C}$, $\mu \neq 0$. Then the Weierstrass product

$$D_{m,M}(\lambda) := \lambda^{m(0)} \prod_{n=1}^{\infty} E_{m,M}(\lambda, -\lambda_n)$$

defines a unique function called the characteristic determinant up to a polinomial of degree less or equal than m $P(\lambda)$

$$D_M(\lambda) = e^{P(\lambda)} D_{m,M}(\lambda).$$

One can generalize almost all these properties to elliptic operators of order d and then Weyl's law traduces:

$$\lambda_n \sim n^{d/m} \frac{(2\pi)^d}{(\mathrm{vol}(M)\Gamma(\frac{m}{d}+1))^{m/d}} \quad n \to \infty.$$

It would be useful too to traduce these properties to the wave equation using $\sqrt{\Delta}$ instead of Δ as we've done in the case of the heat equation.

We focus now in our case: $M = X := \Gamma \setminus \mathbb{H}$ and g given by the line element (see section 2.6)

$$ds^2 = \frac{dx^2 + dy^2 + dr^2}{r^2}$$

where $\Gamma < \mathbf{PSL}_2(\mathbb{C})$ is a cocompact group. We apply the Selberg trace formula to the function $h(1-s^2) = h(\lambda) = e^{-\lambda t} = e^{(s^2-1)t}$. We have that $h(1+s^2) = e^{-(1+s^2)t}$ satisfies the growth conditions of Theorem 4.2. We have

$$g(x) = \frac{1}{\sqrt{4\pi t}} e^{\frac{-x^2}{4t} - t}, \quad \int_{\mathbb{R}} h(1+s^2)s^2 \, ds = \frac{\sqrt{\pi}}{2t\sqrt{t}}$$

$$\implies e^t \theta_X(t) = \frac{v(\mathcal{F})}{(4\pi t)^{\frac{3}{2}}} + Eg(0) + \sum_{\{T\} \text{ lox.}} \Lambda(T) \; g(\log N(T))$$

$$= \frac{v(\mathcal{F})}{(4\pi t)^{\frac{3}{2}}} + \frac{1}{\sqrt{4\pi t}} \left(E + \sum_{\{T\} \text{ lox.}} \Lambda(T) \; e^{-\frac{\ell_T^2}{4t}} \right).$$

Then Weyl's law follows by a tauberian theorem ([31]).

The heat kernel satisfies

$$K_X(t,x,y) = \sum_{\gamma \in \Gamma} K_{\mathbb{H}}(t, \tilde{x}, \gamma \tilde{y})$$

where $\pi(\tilde{x}) = x, \; \pi(\tilde{y}) = y$ with $\pi : \mathbb{H} \to X$ canonical the projection and

$$K_{\mathbb{H}}(t, \tilde{x}, \tilde{y}) = \frac{1}{(4\pi t)^{3/2}} \frac{d_{\mathbb{H}}(\tilde{x}, \tilde{y})}{\sinh d_{\mathbb{H}}(\tilde{x}, \tilde{y})} e^{-t - \frac{d_{\mathbb{H}}(\tilde{x},\tilde{y})^2}{4t}}$$

is the heat kernel on $\mathbb{H}$ ([10]). Let $\ell_\gamma = \ell_\gamma(x,y) = d_{\mathbb{H}}(\tilde{x}, \gamma\tilde{y})$ then for $x \neq y$ the heat kernel reads

$$K_X(t,x,y) = \frac{1}{(4\pi t)^{3/2}} e^{-t} \sum_{\gamma \in \Gamma} \frac{\ell_\gamma}{\sinh \ell_\gamma} e^{-\frac{\ell_\gamma^2}{4t}}.$$

It's not difficult to see that the function $\Phi(s)$ defined in chapter 4 plays a similar role as $D_X(s)$, the difference lies in that $\Phi(s)$ is taken over the shifted eigenvalues s_n (defined in chapter 3). Similar arguments given in [4] give an integral expression of the trace formula for a suitable function h

$$\sum_{n=0}^{\infty} h(\lambda_n) = \frac{1}{2\pi i} \int_C h(1+s^2)\ d\log \Phi(s)$$

where C is the path given in [4] but including the eigenvalues of the interval $[1/2, 1]$.

5.2 From primes to geodesics

Take $M = \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ with metric $ds^2 = dx^2$. Then the Laplace-Beltrami operator is $\Delta = \frac{d^2}{dx^2}$ and its spectrum is given by the sequence $\{n^2\}_{n\geq 0}$ with multiplicity 2 for the non-zero eigenvalues. The eigenfunctions are $\varphi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx},\ k \in \mathbb{Z}$. Then we have

$$D_{\mathbb{S}^1}(s) := \sinh(\pi s) = \frac{1}{2} e^{\pi s}(1 - e^{-2\pi s}) = \pi s \prod_{n=1}^{\infty} \left(1 + \frac{s^2}{n^2}\right)$$

since

$$\frac{d}{ds} \log D_{\mathbb{S}^1}(s) = \pi \coth(\pi s) = \frac{1}{s} + \sum_{k=1}^{\infty} \frac{(2\pi)^{2k}}{(2k)!} B_{2k} s^{2k-1} = \frac{1}{s} - 2\sum_{k=1}^{\infty} (-1)^k \zeta(2k) s^{2k-1}$$

$$= \frac{1}{s} + 2s \sum_{k=0}^{\infty}\sum_{n=1}^{\infty} (-1)^k \frac{s^{2k}}{n^{2(k+1)}} = \frac{1}{s} + 2s \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} (-1)^k \frac{s^{2k}}{n^{2(k+1)}}$$

$$= \frac{1}{s} + 2s \sum_{n=1}^{\infty} \frac{n^{-2}}{1 + \frac{s^2}{n^2}} = \frac{1}{s} + 2s \sum_{n=1}^{\infty} \frac{1}{s^2 + n^2} = \frac{1}{s} + \sum_{n=1}^{\infty} \left(\frac{1}{s+in} + \frac{1}{s-in}\right)$$

for $s \in]0,1[$. The other symmetric functions are

$$K_{\mathbb{S}^1}(t,x,y) = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} e^{-k^2 t} e^{ik(x-y)}, \quad \theta_{\mathbb{S}^1}(t) = 1 + 2\sum_{k\geq 1} e^{-k^2 t},$$

$$\Theta_{\mathbb{S}^1}(t) := 1 + 2\sum_{k\geq 1} e^{-kt}, \quad \zeta_{\mathbb{S}^1}(s) = \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n^{2s}} = 2\zeta(2s)$$

where $\zeta(s)$ is the Riemann zeta function which admits an Euler product expansion (with $\mathbb{P}$ the set of prime numbers)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\in\mathbb{P}} \frac{1}{1-p^{-s}} = \prod_{p\in\mathbb{P}} (1 - e^{-s\log p})^{-1}.$$

We can observe the similarity of $D_{\mathbb{S}^1}(s)$ with the Selberg zeta function $Z(s)$ since $\mathbb{S}^1$ has only one primitive geodesic of length 2π. The Riemann zeta function has not this strong similarity because this would mean that in the given manifold one could have $\log p$ as the length of primitive geodesics and since there are an infinite number of primes the manifold wouldn't be compact (note that all symmetric functions are defined on compact manifolds). But there are some similar results about primes as about primitive geodesics, The Prime Number Theorem and The Prime Geodesic Theorem are an example.

Theorem 5.6. (The Prime Number Theorem)

$$\pi(x) \sim \frac{x}{\log x}$$

where

$$\pi(x) := |\{p \in \mathbb{P} : \ p \leq x\}|.$$

It's not difficult to derive a trace formula in $\mathbb{S}^1$ (called the Poisson summation formula). Take a function $h : \mathbb{C} \to \mathbb{C}$ with Fourier transform g, then for suitable properties for h the trace formula reads

$$\sum_{n \in \mathbb{Z}} h(n) = \frac{1}{2\pi i} \int_C h(is) \ d \log D_{\mathbb{S}^1} = \sum_{m \in \mathbb{Z}} g(2\pi m)$$

where $C = C_+ \cup C_-$ is a path given by the lines $C_+ : \ \Im s = \eta$ and $C_- : \ \Im s = -\eta$ (where C_+ goes from $-\infty$ to ∞ and C_- from ∞ to $-\infty$). The first equality is given by the calculus of residues and the second by an expantion of exponentials

$$d \log \sinh(\pi s) = \pi \coth(\pi s) = \pi \frac{1 + e^{-2\pi s}}{1 - e^{-2\pi s}} = 2\pi \Theta(-2\pi s), \quad \text{for } \Im s > 0,$$

$$d \log \sinh(\pi s) = -2\pi \Theta(2\pi s), \quad \text{for } \Im s < 0.$$

By the change of variable $s \to is$ we obatin

$$\frac{1}{2\pi i} \int_C h(is) \ d \log D_{\mathbb{S}^1} = \frac{1}{2\pi} \int_{-\infty - i\eta}^{+\infty - i\eta} h(s) \Theta(-2\pi i s) \ ds - \frac{1}{2\pi} \int_{-\infty - i\eta}^{+\infty - i\eta} h(s) \Theta(2\pi i s) \ ds$$

if we impose h holomorphic in the strip $|\Im s| \leq \eta$ for $\eta > 0$ we can tend η to zero in the integral to obtain

$$\int_{-\infty - i\eta}^{+\infty - i\eta} h(s) e^{-2\pi i k s} \ ds = \int_{-\infty}^{+\infty} h(s) e^{-2\pi i k s} \ ds = 2\pi g(2\pi k)$$

$$\Longrightarrow \int_C h(is) \ d \log D_{\mathbb{S}^1} = 2\pi i \sum_{m \in \mathbb{Z}} g(2\pi m).$$

Since we have changed the summation order in the process we need h to satisfy $h(s) = O((1 + |s|)^{-1-\delta})$ for some $\delta > 0$ and as $s \to \infty$.

One can obtain easily the Riemann Functional Equation ([29]) for $\Re s < 0$ and extended by meromorphic continuation to $\mathbb{C}$

$$\zeta(s) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}} \left(2 \sin \frac{\pi s}{2}\right) \zeta(1-s).$$

The completed zeta function

$$\Xi(s) := G^{-1}(s)(s-1)\zeta(s),$$

with: $G(s) := \frac{\pi^{s/2}}{2\Gamma(1+s/2)}$,
is an entire function with the same zeros as $\zeta(s)$ except the trivial ones. We also have (by the Functional Equation)

$$\Xi\left(\frac{1}{2}+s\right) = \Xi\left(\frac{1}{2}-s\right) \quad s \in \mathbb{C}..$$

The function Ξ is of order 1 and for any $\varepsilon > 0$ we have $\sum \rho^{-1-\varepsilon} < \infty$ ([29]), where the sum runs over the zeros of Ξ. The Riemann zeros occur in pairs $(\rho, 1-\rho)$ (by the Functional Equation) and we enunmerate all the zeros of Ξ as follows

$$\{\rho_k = \frac{1}{2} \pm i\tau_k\}_{k\geq 1}, \quad \Re\tau_k > 0$$

where we impose that $\{\Re\tau_k\}_{k\geq 1}$ is a non-decreasing sequence. The Riemann Hypothesis states that $\tau_k \in \mathbb{R}$ for all $k \geq 1$. The Functional Equation implies that the zeros of Ξ lie in the strip $0 \leq \Re s \leq 1$. The function Ξ admits the Hadamard factorization

$$\Xi(s) = \Xi(0) e^{\frac{d}{ds}(\log \Xi)(0)s} \prod_\rho \left(1 - \frac{s}{\rho}\right) e^{s/\rho} = e^{\log(2\sqrt{\pi})-1-\gamma/2} \prod_\rho \left(1 - \frac{s}{\rho}\right) e^{s/\rho} \quad \forall s \in \mathbb{C}$$

where γ is the Euler-Mascheroni constant and the product runs over the zeros of Ξ. This gives

$$\frac{\zeta'}{\zeta}(s) = \log(2\sqrt{\pi}) - 1 - \gamma/2 - \frac{1}{s-1} - \frac{1}{2}\psi(1+s/2) + \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

where $\psi(s) = \frac{\Gamma'}{\Gamma}(s)$. Note that

$$\frac{\zeta'}{\zeta}(s) = -\sum_{p\in\mathbb{P}} \sum_{r=1}^{\infty} p^{-sr} \log p = -\sum_{n\geq 2} \Lambda(n) n^{-s}.$$

where $\Lambda(n)$ is the von Mangoldt function defined to be equal to $\log p$ if n is a power of a prime number and 0 otherwise.

Define

$$A(T) := |\{\tau_k : \ \Re\tau_k \leq T\}|$$

then the Riemann-von Mangoldt formula describes the growth of $A(T)$.

Theorem 5.7.

$$A(T) = \frac{T}{2\pi}\left(\log\frac{T}{2\pi} - 1\right) + O(\log T) \quad \text{for } T \to \infty.$$

In the case of the eigenvalues of the Laplace-Beltrami operator we have obtained a trace formula. In number theory, a trace formula can be obtained "Selberg class" Dirichlet series ([29], [27]) is given, the trace formula is called Explicit Formula. The Guinand-Weil formula is an Explicit Formula for the Riemann zeros.

Theorem 5.8. *Let $h : \mathbb{R} \to \mathbb{C}$ be a function and g its Fourier transform. If h is an even and there exist $\delta, \varepsilon > 0$ such that in a strip $|\Im\tau| < 1/2 + \delta$ h extends to an holomorphic function and $h(\tau) = O((1 + |\tau|)^{-1-\varepsilon})$ for $\tau \to \infty$. Then the Explicit Formula reads*

$$\sum_{k=1}^{\infty} h(\tau_k) = h\left(\pm\frac{1}{2}i\right) + \frac{1}{4\pi}\int_{-\infty}^{\infty} h(\tau)\left(\psi\left(\frac{1}{4} + \frac{1}{2}i\tau\right) - \log\pi\right)\, d\tau - \sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}g(\log n).$$

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